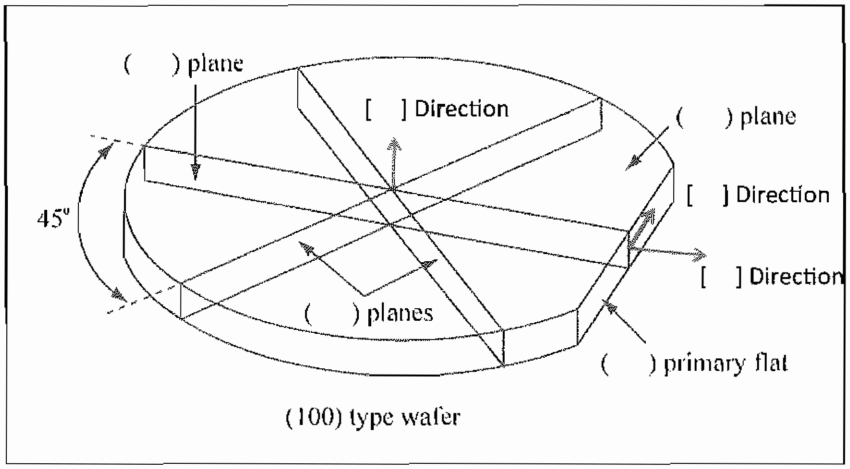
MICRO/NANO ENGINEERING
Doctoral Qualifying Examination, May 2016
Mechanical Engineering Department, Columbia University

Question 1. Fabrication

a) Identify the following on the (100) silicon wafer: (Please redraw in your exam book



b) **Cleaning:** Why is wafer cleaning critical to microfabrication, photolithography, metal deposition, etc.? What does each of the following cleaning processes: RCA-1 (or SC-1) and RCA-2 (SC-2) remove?

c) **Thin Films:**

i.) Briefly describe 3 methods for forming a 300 nm thick oxide (SiO_2) layer on a silicon wafer. Compare and contrast the oxides – e.g. what method produces the highest quality oxide, what process parameters must one take into consideration when selecting a method of forming such an oxide?

ii.) Silicon nitride is another commonly used thin film in MEMS. Why might you use silicon nitride in a MEMS device over silicon dioxide? What is silicon nitride often used for during MEMS fabrication and processing and why? Why might you select silicon nitride for a diaphragm in a MEMS microphone specifically, why not use polysilicon?

d) **Photolithography:** What is the difference between a positive and negative photoresist?

e) **Metal Deposition:** Identify 3 methods for depositing metals. Compare and contrast each method. Give a fabrication/application specific reason for when to select each metal deposition process.

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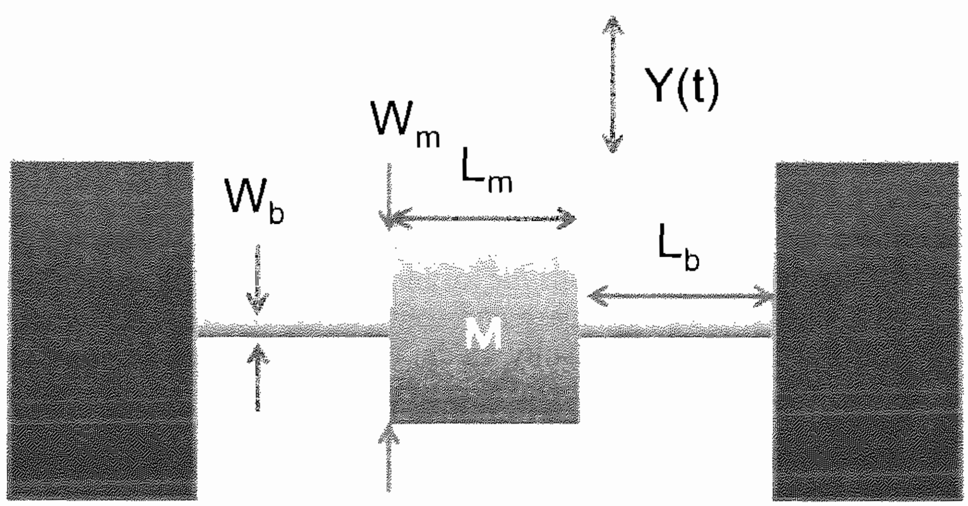
Question 2. Mechanics & Accelerometers

Consider the following structure, assume the dimensions W_m=L_m=L_b= 200 um, W_b = 10 um, and a silicon film made with thickness H= 25 um. Assume single crystal Silicon has a Young's modulus of 180 GPa and a density of 2330 kg/m^3.

a) Calculate the total spring constant of the two cantilevers, assuming they behave as guided beams. *Note the direction of deflection.
b) Calculate the resonance frequency of this system. Neglect the mass of the beams.
c) Calculate the deflection of the mass, Y(t), under an acceleration in the y direction of **a** = 20 g.
d) What is the relative sensitivity to acceleration in the out of plane Z direction?
e) Assuming this structure has a quality factor of 10, what is its equivalent damping ratio?
f) At its resonant frequency and with this quality factor, what is the expected deflection in the y-direction?



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Question 3. Dynamics

MEMS devices are often modeled as a second order system using the following equation:

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + k \cdot x = F(t)$$

where *f(t)* is a excitation force. Assume harmonic excitation in the frequency domain of $F_0e^{(s*t)}$ and a similar output function in the frequency domain $x(t)= X_0e^{(s*t)}$, as we know the output will match the input. Assume that the natural frequency is $\omega_n = \left(\frac{k}{m}\right)^{\frac{1}{2}}$ and the quality factor Q is defined as $Q = \frac{\omega_n \cdot m}{b}$.

a. Solve this differential equation for X_0. Keep the result in the frequency domain.
b. Find the expression for the amplitude response of the resulting transfer function.
c. Draw a representative amplitude (also called magnitude) frequency response of this second-order system. Qualitatively illustrate the amplitude response of a MEMS sensor with a quality factor of .1 and 10.
d. Examine the amplitude frequency response plot (Bode Plot) from (c). At low frequencies, approaching zero Hertz or DC what do you observe?
e. If you are using a piezoelectric sensing mechanism with such a second-order system, what do you expect to happen, as you approach zero Hertz or DC?
f. If you are using a piezoresistive sensing mechanism with such a second-order system, what do you expect to happen, as you approach zero Hertz or DC?

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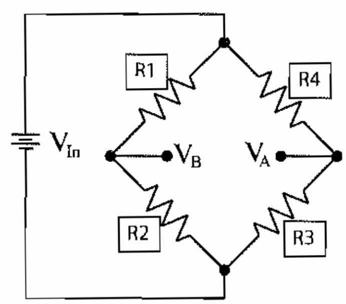
Question 4. MEMS Resonator & Piezoresistive Sensing

a. **MEMS Resonator:** A micromechanical resonator is vibrating at a frequency of $f_0 = 100kHz$ with a vibration amplitude of $x_0 = 8\ nm$. Vibrations are measured using the capacitive rate-of-change measurement with a parallel plate capacitor that has area $A = 1{,}000\ \mu m^2$ and the gap $d = 1\ \mu m$. Calculate the motional current if the bias votltage over the capacitor is $V_{dc} = 5\ V$.
$\epsilon = 8.854 \times 10^{-12} F/m$

b. **Capacitive Sensing:** Why are differential measurements used in capacitive sensing?

c. **Wheatstone Bridge - Piezoresistance Sensing:**



Solve the Wheatstone bridge above for V_0/V_{in} as a function of R_1, R_2, R_3, R_4, where $V_0 = V_A - V_B$. You must show your work.

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MICRO/NANO ENGINEERING
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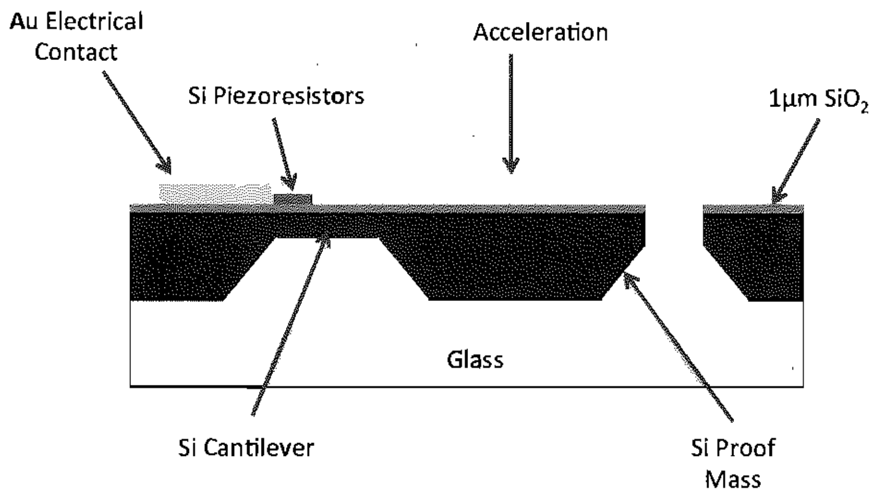
Question 1.



The piezoresistive accelerometer shown above has a proof mass that has a length, L = 1mm and a width, W=1mm and a thickness, H = 400 μm. The mass is anchored via a cantilever beam that has a length, L_b = 250 μm a width, w_b = 80 μm and a thickness, t_b = 10 μm. At the high-stress location on the beam, four p-type silicon piezoresistive elements are fabricated with a doping concentration of $5\times10^{16} cm^{-3}$. Below doping concentrations of 10^{17} cm^{-3}, the following longitudinal and transverse piezoresistive coefficients apply to a p-type (100) wafer:

p-Type Silicon	[100] Direction	[110] Direction
π_l [10^{-11} 1/Pa]	6.6	71.8
π_t [10^{-11} 1/Pa]	-1.1	-66.3

Assuming the following material properties and ignoring the SiO_2 layer, solve for the following:

	Single Crystal Silicon
E [GPa]	170
ρ [kg/m^3]	2330
μ_p [cm^2/V*s]	450

a.) Derive the basic equation for the maximum stress where the Si piezoresistive elements are located as a function of the width, length, height of the beam and the length of the mass.

Question 1. (Cont.)

b.) What is the maximum stress in the Si piezoresistive elements at 1g of input acceleration?

c.) How should the four Si p-type piezoresistors be oriented along the surface of the cantilever beam in order to maximize the output of the device? Which piezoresistive coefficients are applicable?

d.) The conductivity of Si is expressed as $\sigma = e \cdot \mu_p \cdot p$, where μ is the mobility of holes, p is the carrier concentration and e is the charge of an electron, $1.602 \times 10^{-19}\ C$. If we would like to design a sensor such that each arm of the Wheatstone bridge has a resistance of 10 kΩ and the device thickness is controlled to 1 μm, what length/width ratio or number of squares are needed? What is the sheet resistance of this device in units of Ω/square?

e.) Based upon the stress from 1g of acceleration (b), what is the expected change in resistance of each arm of the bridge? (Only consider the max stress along the beam).

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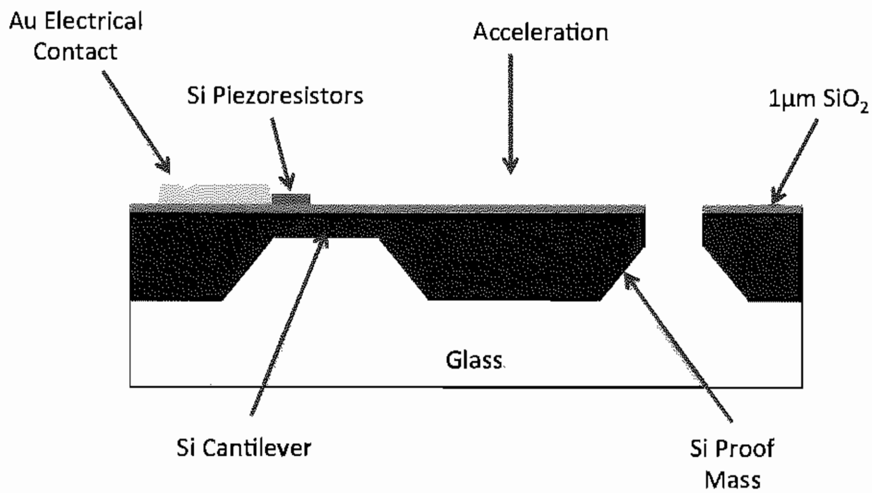
Question 2.



MEMS batch fabrication processes have enabled mass production of MEMS devices as hundreds to thousands of devices can be made simultaneously using minimal material. Layout a full fabrication process flow for making the piezoresistive cantilever beam with mass accelerometer shown above. Be sure to note cleaning steps and draw basic photolithography masks along with cross sections of each step. Provide arrows indicating components. Please hatch cross sectional images so that different materials are easily identified. For simplicity, only include a single p-type Si piezoresitor, made from a (100) wafer that is aligned along the longitudinal stress at the max stress location on the cantilever. Be sure to draw a top view of the resistor pattern mask. You can start with a Silicon on Insulator (SOI) wafer to simplify the processing, i.e. Silicon – oxide – silicon of the desired orientation and doping.

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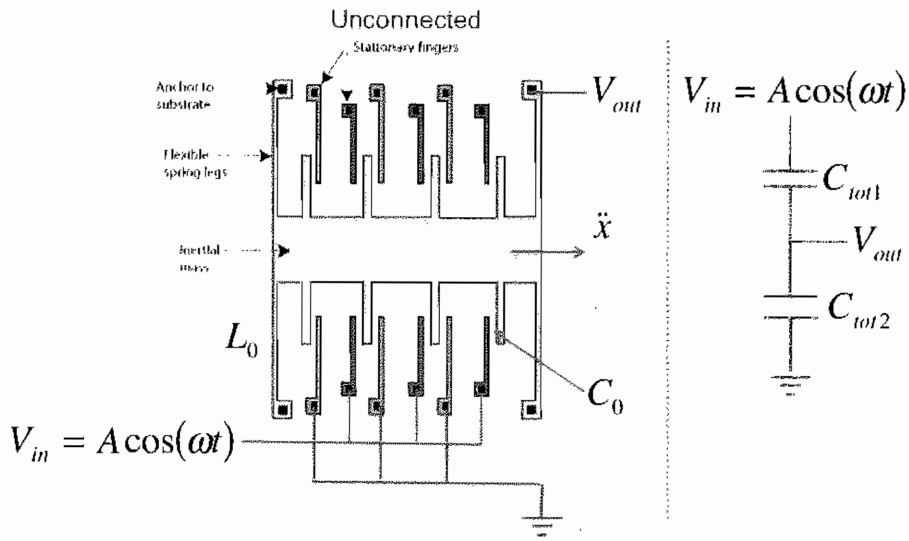
Question 3.



Consider the capacitive accelerometer made out of poly-Silicon illustrated above. Please take careful note of the electrical wiring.

	Poly-Silicon
E [GPa]	160
ρ [kg/m^3]	2330

a. This capacitive accelerometer is suspended via four beams. What special type of beams supports the proof mass of this accelerometer and how do you identify them? Are these four beams in parallel or series and how do you know? What is the total spring constant for the 4 beams if the beams are each $L0 = 100$ μm long, with a width and thickness of 2 μm.

b. Why do we apply a sinusoidal input voltage to this device? If each gap on the comb detector has capacitance $C0 = 0.1$ pF and the device is wired up as shown, determine the equation for the output voltage *Vout* as a function of the input voltage, V_{in}.

Question 3. (Cont.)

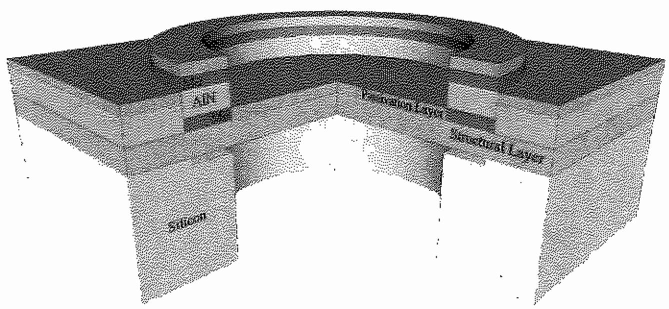
c. If Ctot1 =Ctot2= 0.2 pF and the initial gap, d_0 = 2 μm, what is the total sensitivity of this device for a frequency much lower than the resonance frequency? Assume the input voltage has a peak-to-peak value of 2 V (A=1V) and a proof mass of 0.2nKg.

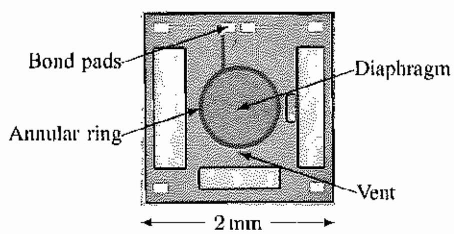
MICRO/NANO ENGINEERING
Doctoral Qualifying Examination, January 2016
Mechanical Engineering Department, Columbia University

Question 4.

Piezoelectric MEMS Microphone

Piezoelectric MEMS microphones are currently an active area of research due to the low power consumption of these devices and good performance. Examine the cross-sectional view (top) and an optical microscope image (bottom) of the MEMS piezoelectric microphone below:





Williams, et. al. J. MEMS. 2012.

Assume the structural layer in the sensor above is Silicon Nitride with a Young's modulus of 380GPa, density of 3100 kg/m3, and a Poisson's ratio of .22. The piezoelectric material used is AlN. The AlN has molybdenum electrodes on the top and bottom surfaces to measure the electric field generated by the piezoelectric under dynamic stresses generated by sound (dynamic pressure). We want to design the above sensor to meet the following design considerations:

70 G

Metric	*MEMS Requirement*
Diaphragm Diameter	600μm
Diaphragm Thickness, t	2.15 μm
Piezoelectric Annulus	r_1=300 μm to r_2=340 μm
Full-Scale Dyn. Pressure	56.5 Pa Peak-to-Peak (~120 dB SPL, 20Pa rms)
Bandwidth	20Hz to 20kHz
Sensitivity w/ Charge Amp	100 μV/Pa

Based upon the dimensions for this silicon nitride membrane listed above and assuming a pretension within this **membrane** of 2MPa, the center point deflection of this membrane at the full-scale, peak-to-peak input dynamic pressure of 56.5 Pa can be calculated to be ~265 *nm*. This is achieved using the equation below for predicting the deflection of membrane with pretension.

$$P = \frac{8Yt}{3a^4}\delta^3 + \frac{4\sigma_0 t}{a^2}\delta$$

where $Y = \frac{E}{(1-\nu)}$ is the biaxial modulus, *a* is the radius of the membrane, *P* is the input pressure, *t* is the thickness of the membrane and δ is the center point deflection of the membrane.

a.) Using this center point deflection of ~265 *nm* at the full-scale pressure of 56.5Pa, peak-to-peak, find the tensile strain within the membrane following the equation below:

$$\varepsilon = \frac{2\delta^2}{3a^2} + \frac{\sigma_0}{Y}$$

b.) Based upon the strain within the membrane, which piezoelectric coefficient should you select using the table below? Using this coefficient, predict the motional current generated by a dynamic strain created by a dynamic pressure. Leave your results in units of *C*s* or *C* jω*.

AlN Piezoelectric Coefficients

e_{33}	e_{31}
1.55 C/m²	-0.48 C/m²

Question 4. (Cont.)

c.) Using your motional current, design a transimpedance amplifier that converts this current to a voltage and eliminates the dependence of the generated current upon the frequency of the input sound. Based upon the required sensitivity of 100 μV/Pa, what is the size of the feedback capacitor that you should select? Hint: The motional current you calculated is for the full-scale input pressure of 56.5Pa peak-to-peak or 20Pa rms.

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2014 ME PhD Qualifying Exam: MEMS questions

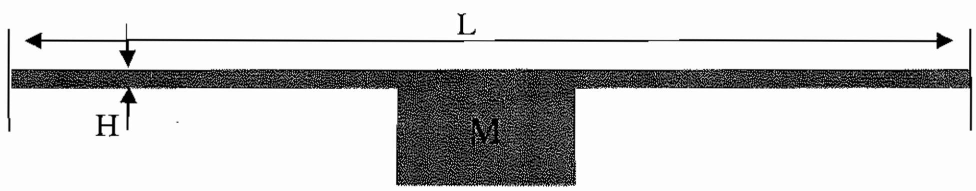
Question 1:

a. [*33 points*] An electrostatically actuated cantilever mirror device is observed to "snap down" to the substrate beyond a threshold voltage applied between the cantilever and the substrate. Describe the electrostatics process and explain why this happens.

b. [*67 points*] A double-clamped silicon beam, shown in the below Figure, is subjected to an acceleration of 55.195 g, where g = 10 m/s^2. Assume the acceleration gives a center-loading to the double-clamped beam, where the load-deflection characteristics (derived in class from energy methods, principle of virtual work, and trial solutions) is

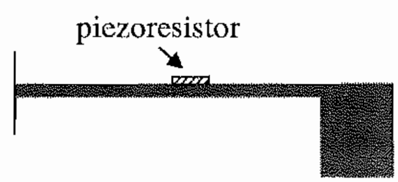
$$F = \left(\frac{\pi^4}{6}\right)\left[\frac{EWH^3}{L^3}\right]c + \left(\frac{\pi^4}{8}\right)\left[\frac{EWH}{L^3}\right]c^3$$

The double-clamped beam has designed parameters L = 1000 um; W = 5 um; H = 2 um; E = 170 GPa. The center-mass M is 1×10^{-10} kg.

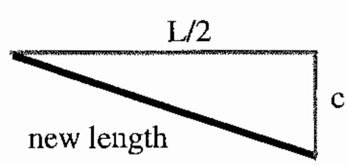


(i) For small deflections (c < H), calculate the deflection of the center of the beam. *[35 points]*

(ii) A p-type piezoresistor, with gauge factor of 200 and nominal resistance R of 10 kΩ, is placed at the location *L/4*. Assuming the strain is uniform along the deflected beam (poor assumption but in the interest of time for this question), calculate the change in the nominal resistance. *[32 points]*



Use the below diagram to calculate the assumed strain:

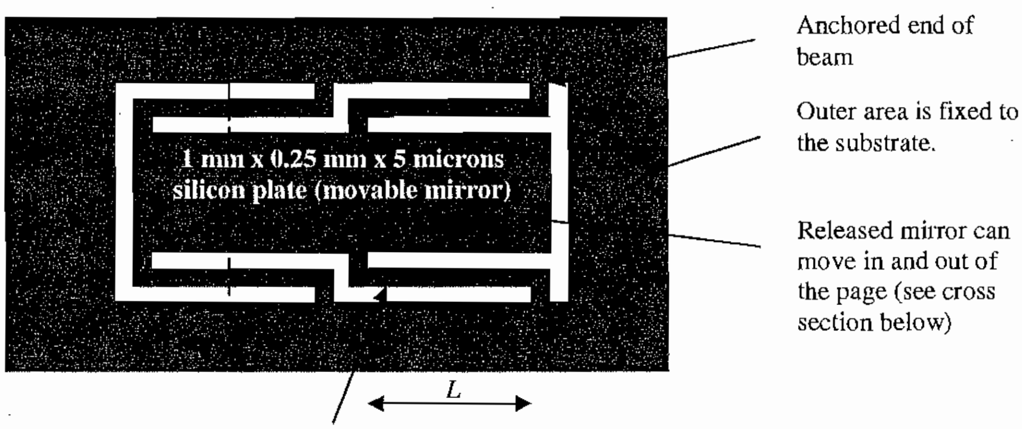


Question 2:

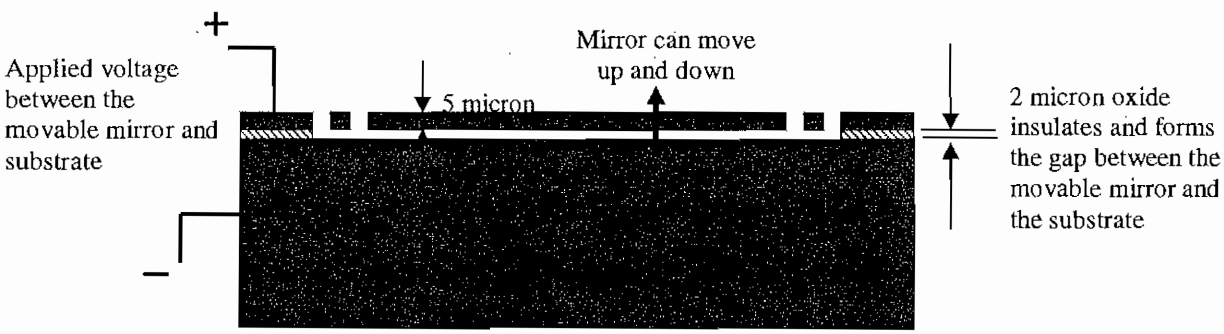
a. [*33 points*] Draw the lumped element design model of squeezed film damping.

b. [*67 points*] Consider the 1 mm x 0.25 mm reflection mode optical phase modulator shown below, designed using a 5 μm thick single crystal silicon wafer bonded to an oxidized silicon substrate. Light reflected from the top surface of the plate can be phase shifted by moving the plate vertically by applying a voltage between the top and bottom silicon layers. The oxide layer is 2.0 μm thick, and acts as an insulator between the thin top layer and the substrate. The four support springs are 0.5 mm long and 50 μm wide.

(i) Calculate the total spring constant of the support spring system and the resonant frequency of the device (in Hz), neglecting the mass of springs. *[33 points]*
(ii) Considering squeeze film damping under the plate, estimate the quality factor, *Q*. Suggest one approach to reduce the damping and therefore increase *Q*. *[34 points]*



0.5 mm long, 50 micron wide, 5 micron thick silicon beams



Cross-section A-A

Note: The spring constant for a single beam in this problem is given by:

$$k = \frac{F}{v} = \frac{12EI}{(L)^3}; I = \frac{Wt^3}{12}$$

where F is the load taken by one beam, v is the mirror displacement, E is Young's Modulus (160 GPa for Si), W is the beam width, t is the beam thickness, and L is the length of the beam.

The density of silicon is 2300 kg/m^3
The permittivity of air (or free space) is ε=8.85×10^{-12} F/m
The viscosity of air at ambient pressure and temperature is μ=2×10^{-5} Pa·s

Other potentially useful formulas. The meaning of each symbol should be clear:

$$F_{normal} = \frac{\varepsilon A V^2}{2d^2}; F_{lateral} = \frac{\varepsilon W V^2}{2d};\ C = \frac{\varepsilon A}{d} \text{ for a parallel plates}$$

$$\omega_o = \sqrt{\frac{k}{m}};\ Q = \frac{\omega_o m}{b} \text{ for general 2}^{\text{nd}} \text{ order dynamic systems}$$

$$b = \frac{96\mu L W^3}{\pi^4 d^3} \text{ for normal motion of parallel plates}$$

$$V_{pi} = \sqrt{\frac{8kd_0{}^3}{27\varepsilon A}}, \text{ where } d_0 \text{ is the gap spacing at 0V.}$$

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Question 3:

a. [*33 points*] Describe the differences between piezoelectric and piezoresistive sensing. Describe how and why a Wheatstone bridge is used in silicon piezoresistive sensors.

b. [*33 points*] Describe the respective electrical resistor-inductor-capacitor lumped elements (if any) for the mechanical and thermal domains. What is the generalized effort and flow in each of the two cases?

c. [*34 points*] Consider the design of a silicon-based pressure sensor using a silicon membrane and piezoresistive sensing.

Note: for a square membrane, $$\frac{PL^4}{Eh^4} = \frac{\pi^4}{6(1-\nu^2)}\left(\frac{y}{h}\right)$$

where, P = applied pressure, in Pa; L = diaphragm side length, in m; E = Young's Modulus (1.7E+11 Pa for Si), in Pa; h = diaphragm thickness, in m; ν = Poisson's ratio (0.2 for Si); y = deflection at the diaphragm's center, in m.

(i) Calculate the thickness for a 1 mm x 1 mm silicon square membrane designed to exhibit a center deflection to thickness ratio of 1% at 10^5 Pa applied pressure. *[17 points]*

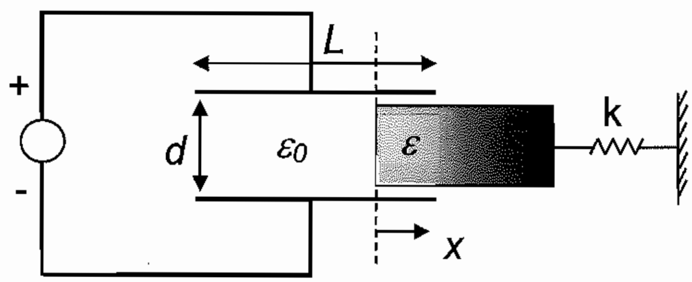
(ii) Roughly estimate the strain in this membrane at this 1% deflection (1% deflection to thickness ratio) by calculating the increased diagonal length along the surface of the membrane assuming uniform strain. Use this strain value along with a reasonable gage factor to estimate the piezoresistance required to cause a 10 ohm resistance change. *[17 points]*

Microelectromechanical Systems

Doctoral Qualifying Examination, January 2013 Department of Mechanical Engineering

Question 1 (30 points total). Consider the electrostatic microactuator below. The dielectric material slides laterally between two parallel plate capacitors. The dielectric material has a permittivity of $\varepsilon = \varepsilon_r \varepsilon_o$.

The gap between the two plates is *d*, the length of each plate is *L*, and the width (out-of-plane, normal to the cross-sectional drawing) is *W*. A linear spring (*F* = *ke,* where *e* is spring extension) is attached to the dielectric.



Useful formulas: $C = \dfrac{W\int \varepsilon dx}{d}$

$\mathbf{W} = \dfrac{CV^2}{2}$, where **W** is the co-energy in a capacitance plate

$F = \left.\dfrac{\partial \mathbf{W}}{\partial x}\right|_V$

(a) Find the expression for the electrostatic force of the actuator at constant voltage *V*. Hint: The system can be modeled as two capacitances *in parallel*. *[10 points]*

(b) Suppose now if $\varepsilon = \varepsilon_r \varepsilon_o\ (1 + x^n)$. Find the expression for the electrostatic force of the actuator at constant voltage *V*. *[10 points]*

(c) If n = 2: will the microsystem exhibit pull-in? Why?

If n = 0.5: will the microsystem exhibit pull-in? Why?

[10 points]

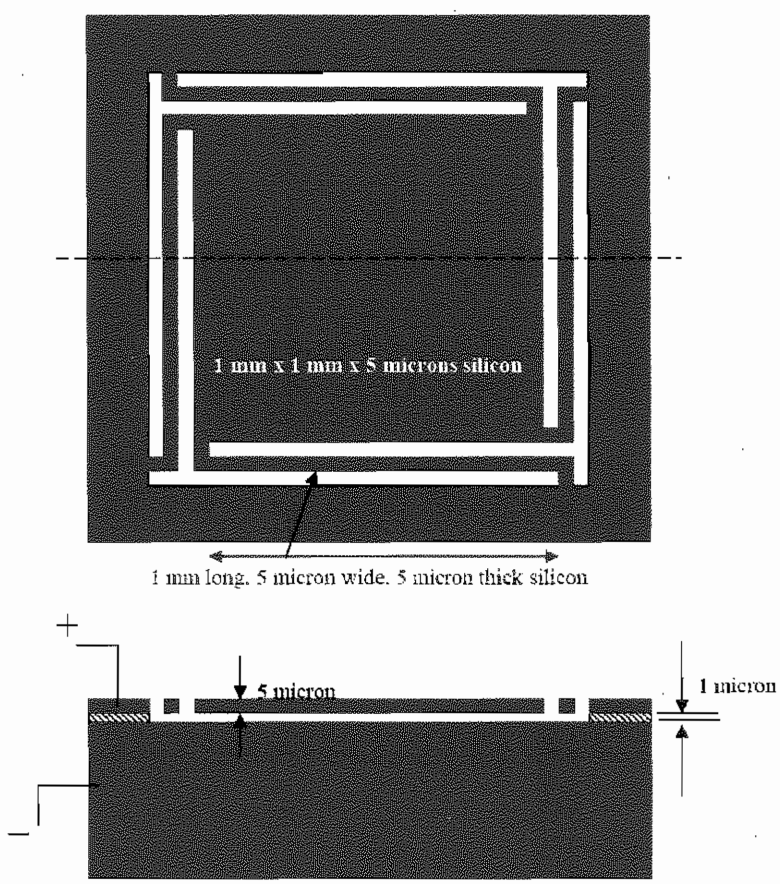
Question 2 (30 points total).

1. [***15 points***] An electrostatically actuated cantilever micromirror device is observed to "snap down" to the substrate beyond a threshold voltage applied between the cantilever and the substrate. Explain why this happens.

2. [***15 points***] Briefly describe 2 fabrication methods for forming a 2 μm thick, 1 mm^2 membrane. List the advantages and disadvantages of each approach for MEMS pressure sensor applications.

Question 3 (30 points total). Consider the 1 mm x 1 mm reflection mode optical phase modulator shown below, designed using a 5 μm thick single crystal silicon wafer bonded to an oxidized silicon substrate. The oxide layer is 1.0 μm thick. Light reflected from the top surface of the plate can be phase shifted by moving the plate vertically by applying a voltage between the top and bottom silicon layers. The four support springs are 1 mm long and 5 μm wide.

a) Calculate the total spring constant of the support spring system and the resonant frequency of the device, neglecting the mass of springs. *[9 points]*
b) Assuming that the plate is held level with the ground, calculate the vertical deflection of the device under its own weight. *[6 points]*
c) Assuming a positive voltage on the plate with respect to the substrate, calculate the pull-in voltage for this device, neglecting the effects of gravity. *[6 points]*
d) Determine the minimum oxide thickness required to prevent breakdown before pull-in, assuming a x2 safety factor for the maximum allowable oxide electric field. Assume an oxide breakdown field strength of 8 MV/cm. *[6 points]*
e) Will reversing the polarity in part d) cause the plate to be repelled from the substrate? Why o why not? *[3 points]*



Note: The spring constant for a single beam is given by:

$$k = \frac{F}{c} = \left(\frac{\pi^4}{6}\right)\left[\frac{EWH^3}{(L)^3}\right]$$

where *F* is the applied load, *c* is the central beam displacement, *E* is Young's Modulus (1.6E+11 Pa for Si), *W* is the beam width, *H* is the beam thickness, and *L* is the length of the beam. The density of silicon is 2.3E3 kg/m^3, and the permittivity of air is 8.85E-12 F/m.

Other useful formulas. The meaning of each symbol should be clear:

$$F = \frac{Q^2}{2\varepsilon A}$$ for a parallel plate capacitor

$$F = kx$$

$$Q = CV$$

$$C = \frac{\varepsilon A}{d}$$

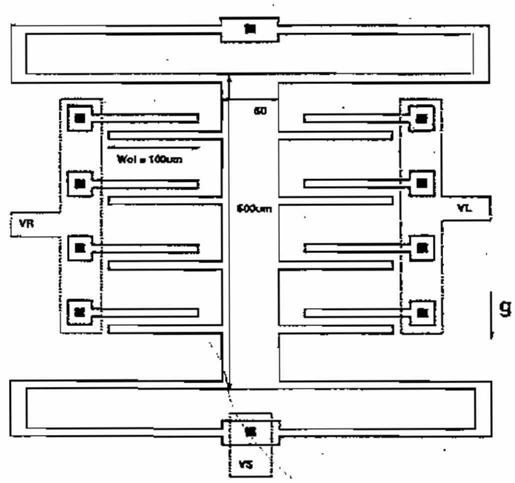
$$\omega_o = \sqrt{\frac{k}{m}}$$

$$V_{pi} = \sqrt{\frac{8kd_0^{\ 3}}{27\varepsilon A}}$$, where d_0 is the gap spacing at 0V.

2012 ME PhD Qualifying Exam: MEMS questions

Question 1:
a. [*33 points*] List the differences between Poiseuille flow and Couette flow in microchannels. Why can we use such Poiseuille or Couette flow approximations, without having to solve the more-involved Reynolds equation, in MEMS devices?

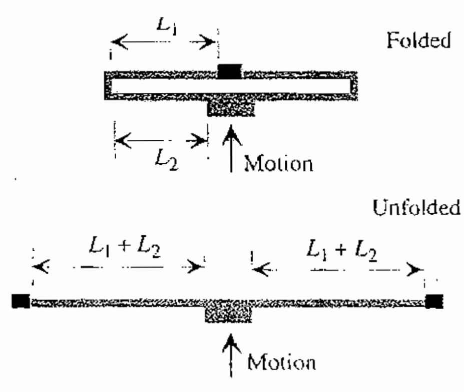
b. [67 points] Consider the accelerometer shown below, designed using a 2 μm thick layer of polysilicon. The proof mass is 50 by 500 μm. The support springs are 200 μm long and 2 μm wide. The variable capacitor fingers have an overlap length of 100 μm and a nominal gap of 2 μm.



b.i. Calculate the total spring constant of the support spring system. (34 points)

b.ii. Assume that we connect VR and VL together and apply a voltage between them and VS. What voltage is required to bring the proof mass back to its rest (0-g) position while it is in a 1-g vertical field. (33 points)

Note:



The above two situations have the same spring constant given by:

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$$k = \frac{F}{c} = \left(\frac{\pi^4}{6}\right)\left[\frac{EWH^3}{(2L)^3}\right]$$

where F is the applied load, c is the central beam displacement, E is Young's Modulus (1.6×10^{11} Pa for Si), W is the beam width, H is the beam thickness, and $2L$ is the total length of the combined unfolded springs. The density of silicon is 2300 kg/m^3, and the permittivity of air is 8.85×10^{-12} F/m.

Other useful formulas. The meaning of each symbol should be clear:

$$F = \frac{Q^2}{2\varepsilon A} \text{ for a parallel plate capacitor}$$

$$F = kx$$

$$Q = CV$$

$$C = \frac{\varepsilon A}{d}$$

$$\omega_o = \sqrt{\frac{k}{m}}$$

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2012 ME PhD Qualifying Exam: MEMS questions

Question 3:

Consider one of the doubly-clamped beams in the Grating Light Valve shown below. Assume squeeze-film damping can be neglected (such as when packaged in vacuum).

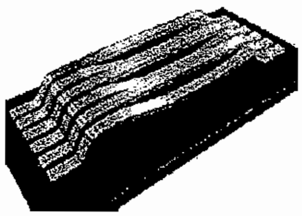

Figure 1. Schematic of a Grating Light Valve with doubly-clamped silicon nitride beams with capacitance plate actuators underneath. [from Sony; Solgaard and Bloom group (Stanford)]

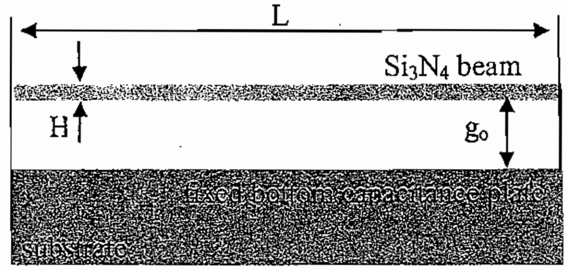

Figure 2. Dimensions of silicon nitride beam: L = 50 um; H = 1 um; W (width of beam; into plane of paper) = 2 um; g_o = 1 um. Silicon nitride has Young's modulus E = 300 GPa, residual stress σ_o= 6 GPa and a density ρ of 3100 kg/m^3.

The dimensions of each silicon nitride beam are shown in the captions of Figure 2. There is residual stress in the beam during the deposition process. The force-deflection characteristic of a doubly-clamped beam is given by:

$$F = \left(\frac{\pi^2}{2}\right)\left[\frac{\sigma_o WH}{L}\right]c + \left(\frac{\pi^4}{6}\right)\left[\frac{EWH^3}{L^3}\right]c + \left(\frac{\pi^4}{8}\right)\left[\frac{EWH}{L^3}\right]c^3$$

The permittivity of air ε_o is 8.85×10^{-12} F/m.

a. [*33 points*] For small deflections ($c < H$), what is the center deflection if we assume a point load of 200 uN at the center of the doubly-clamped beam?

b. [*45 points*] The capacitance actuation plates under the beam acts as a distributed load rather than a point load. Therefore, instead of using the nonlinear spring described above for a point load, an effective stiffness k_{eff} (in N/m) is given by:

$$k_{eff} = \left(\frac{8\sigma_o WH}{L}\right)\left(\frac{1}{1+2\left(\frac{1-\cosh(u)}{u\sinh(u)}\right)}\right)$$

where $u = \left(\sqrt{\frac{3\sigma_o L^2}{EH^2}}\right)$ and $\cosh(u) = \frac{e^u + e^{-u}}{2}$ and $\sinh(u) = \frac{e^u - e^{-u}}{2}$. What is the pull-in voltage for a single beam in the Grating Light Valve?

c. [*22 points*] An estimate of the resonant frequency is given by: $\omega_o = \frac{\pi}{L}\sqrt{\frac{4\sigma_o}{3\rho}}$. What the equivalent lumped mass of the doubly-clamped beam?