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Doctoral Qualifying Examination, January 2015
Mechanical Engineering Department, Columbia University
MATHEMATICS

Problem 1

Solve the following partial differential equation,

$$\frac{\partial^2 f}{\partial x^2} - \frac{\partial f}{\partial t} = 1, \qquad 0 \le x \le 1, \quad 0 \le t$$

subject to the boundary conditions

$$f(0,t) = 0, \qquad \left.\frac{\partial f}{\partial x}\right|_{\substack{\cancel{x=0} \\ x=1}} = 0$$

and the initial condition $f(x,0) = 0$.

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MATHEMATICS

Problem 2

Derive the component form of the gradient operator $\operatorname{grad}\mathbf{v}$ of a vector $\mathbf{v} = v_r\mathbf{e}_r + v_\theta\mathbf{e}_\theta + v_z\mathbf{e}_z$ in cylindrical coordinates (r,θ,z), starting from the definition of the gradient operator, $d\mathbf{v} = (\operatorname{grad}\mathbf{v})\cdot d\mathbf{r}$.

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MATHEMATICS

Problem 3

Consider a square matrix having the partitioned form

$$C = \begin{bmatrix} A_1 & 0 \\ B_1 & A_2 \end{bmatrix}$$

where A_1 us an $n \times n$ matrix, A_2 is an $m \times m$ matrix, B_1 is an $m \times n$ matrix, and the 0 is a matrix of zeros of appropriate dimension. Both A_1 and A_2 are diagonalizable by similarity transformations.

(A) Prove that the eigenvalues of matrix C are the eigenvalues of matrix A_1 together with the eigenvalues of matrix A_2. Give a rigorous mathematical proof that applies in general.

(B) In terms of the eigenvectors of the matrices on the diagonal of C, give some eigenvectors of matrix C.

(C) Give a simple counter example that shows it may not be possible to diagonalize matrix C.

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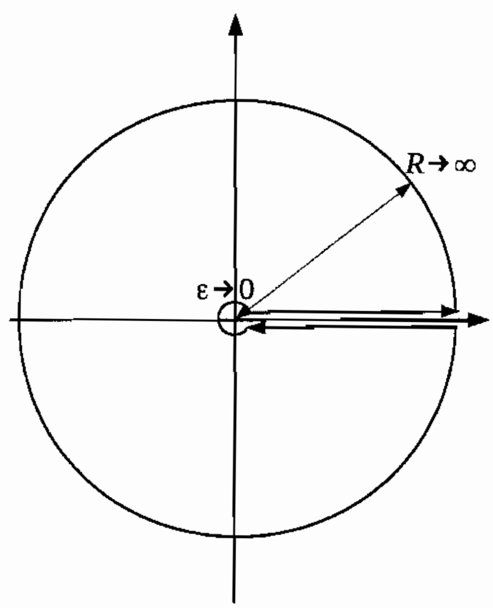
MATHEMATICS

Problem 4

Use complex variable methods to verify that that

$$\int_0^\infty \frac{\log x}{\left(x^2+2x\alpha\cos\tau+\alpha^2\right)}dx=\frac{\tau\log\alpha}{\alpha\sin\tau} \quad \text{when} \quad \alpha>0 \quad \text{and} \quad 0<\tau<\pi.$$

To find the integral $\int_0^\infty dx \log(x) f(x)$, integrate the function $[\log(x)]^2 f(x)$ along the contour shown below. The radius of the smaller circle $\varepsilon\to 0$. The radius of the larger circle $R\to\infty$.



Doctoral Qualifying Examination, January 2015

Mechanical Engineering Department, Columbia University

MATHEMATICS

Problem 5

Use Laplace transforms to solve the following integral equation:

$$y(t)=t+\int_0^t d\tau\, y(\tau)\, e^{(t-\tau)}$$

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MATHEMATICS

Problem 6

Solve the Ordinary Differential Equation

$$\sin x \frac{dy}{dx} + y \cos x - x^2 = 0$$

subject to the condition

$$y(\pi/2) = 1.$$

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Mechanical Engineering Department, Columbia University
MATHEMATICS

Problem 1

Solve the following partial differential equation over a rectangular domain,

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=-1, \qquad -a\le x\le a, \qquad -b\le y\le b$$

for $f(x,y)$, subject to the boundary conditions

$$f(\pm a,y)=0, \quad f(x,\pm b)=0$$

Hint: f is symmetric with respect to x and y axes. Expand the right-hand-side of the PDE as a Fourier cosine series in x.

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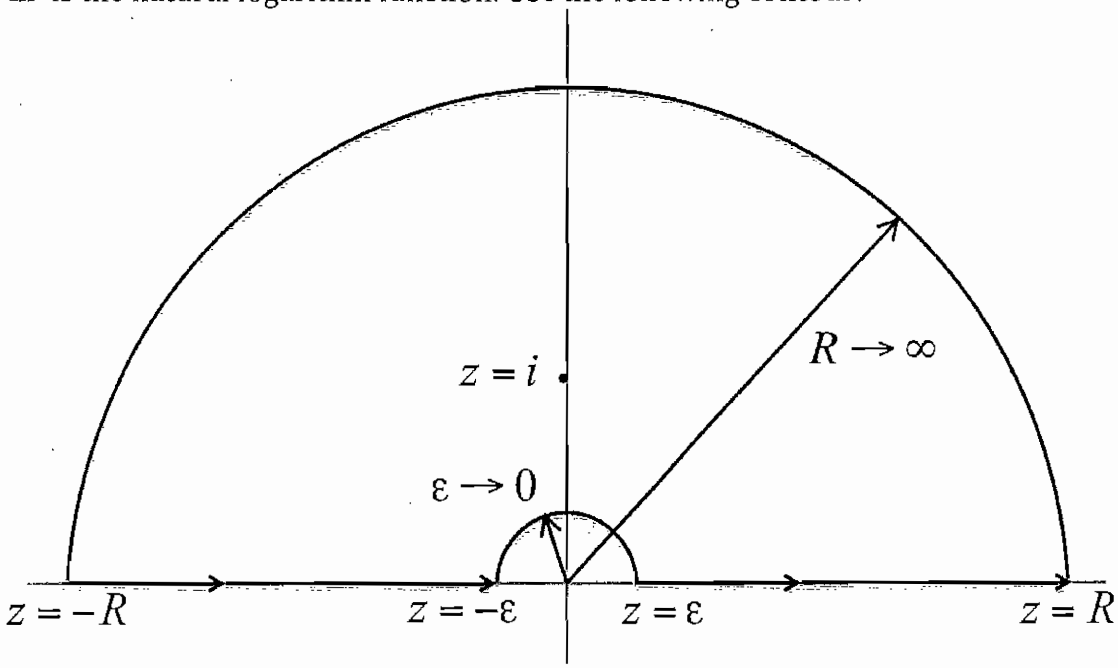
MATHEMATICS

Problem 2

Use contour integration to show that $\int_0^\infty \frac{(\ln x)^2}{1+x^2}dx = \frac{\pi^3}{8}$. In passing, also show that

$\int_0^\infty \frac{\ln x}{1+x^2}dx = 0$.

"ln"is the natural logarithm function. Use the following contour:



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MATHEMATICS

Problem 3

Use Parseval's identity for the Fourier cosine transforms to show that

$$\int_0^\infty dk \frac{p}{p^2+k^2} k^{-n} = (-1)^{n/2} \frac{\pi}{2} p^{-n}$$ when n is even.

For Fourier cosine transform defined as: $F_c(k) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos kx \, dx$, Parseval's identity is given by $\int_0^\infty F_c(k) G_c(k) \, dk = \int_0^\infty f(x) g(x) \, dx$. Choose the functions appropriately to show the identity.

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MATHEMATICS

Problem 4

Obtain the eigenvalues, λ_n, and eigenfunctions, $y_n(x)$, for all possible n for the equation

$$y'' + 4y' + (\lambda^2 + 4)y = 0$$

subject to the conditions

$$2y(0) + y'(0) = 0$$

and

$$y'(1) = 0.$$

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MATHEMATICS

Problem 5
Solve the ordinary differential equation

$$y'' + \lambda^2 y = \sin(x)$$

subject to $y(0) = 0$ and $y'(0) = 0$, where λ is an arbitrary constant.

Show that the general solution breaks down when $\lambda = 1$.

Take the limit of the general solution as $\lambda \to 1$ and demonstrate that it reduces to the solution of $y'' + y = \sin(x)$ subject to subject to $y(0) = 0$ and $y'(0) = 0$.

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MATHEMATICS

Problem 6

Consider a scalar function of a matrix B, $J(B)$. Let the components of matrix B be $b_{\alpha,\beta}$ where is the row number and β is the column number. Define the derivative of scalar $J(B)$ with respect to matrix B as the matrix whose α,β component is $\partial J(B)/\partial b_{\alpha,\beta}$.

(1) Define the trace of a matrix.

(20 Let B be an $m \times n$ matrix and A an $m \times m$ matrix. Show that

$$\partial[tr(B^T AB)]/\partial B = (A + A^T)B$$

Make a proof, not just some example of it working, and explain all of the logic.

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MATHEMATICS

Problem 1

a) Find the Fourier transform of $f(x)=\begin{cases} 1 & |x|<a \\ 0 & |x|>a \end{cases}$, where the Fourier transform of $f(x)$ is defined as $F(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{i\omega x}\,dx$.

b) Use this result to evaluate $\int_{-\infty}^{\infty}\frac{\sin\omega a\cos\omega x}{\omega}d\omega$.

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MATHEMATICS

Problem 2

Use complex variable methods to verify that

$$\int_0^\infty \frac{x\sin\beta x}{\left(1+x^2\right)\left(\alpha^2+x^2\right)}dx=\frac{\pi}{2}\left(\frac{e^{-\beta}-e^{-\alpha\beta}}{\alpha^2-1}\right)$$

when $\alpha,\beta>0$.

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MATHEMATICS

Problem 3

Find the general solution valid near $x = 0$ for

$$x^2 \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + \left(2 - x^2\right) y = 0 .$$

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MATHEMATICS

Problem 4

a) Determine the dimensions of the largest rectangular parallelepiped that can be inscribed in a hemisphere of radius a.

b) If $I(\alpha)=\int_{\alpha}^{\alpha^2} \frac{\sin \alpha x}{x} dx$, find $\frac{dI}{d\alpha}$.

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MATHEMATICS

Problem 5

Find the solution $u(r,t)$ of the partial differential equation

$$u_{tt} = c^2\left(u_{rr} + \frac{1}{r}u_r\right),\ 0 \le r \le R$$

subject to the boundary condition

$$u(R,t) = 0$$

and initial conditions

$$u(r,0) = R^2 - r^2$$
$$u_t(r,0) = 0$$

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MATHEMATICS

Problem 6

Consider the Liapunov equation

$$A^T P + PA = Q$$

Consider all 2 by 2 matrices.
Q is a symmetric matrix
A is a diagonalizable matrix with eigenvalues λ_1, λ_2
P is the unknown.

a) Prove that P is symmetric.
b) By diagonalizing A, develop the mathematical solution of the equation.
c) Under what conditions on matrix A is there a solution to the Liapunov equation?

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MATHEMATICS

Problem 1

Find the Fourier transform of $e^{-(x^2-2xt+t^2)/2}$, where the Fourier transform of $f(x)$ is defined as

$$F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx$$

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MATHEMATICS

Problem 2

Assuming that $-1 \leq a \leq 1$, find the integral

$$\int_0^{2\pi} \frac{1}{1 + a\ \cos\theta}\, d\theta$$

using contour integration.

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MATHEMATICS

Problem 3

The equation governing the displacement $y(x)$ of a beam (positive displacement is up) on an elastic foundation is

$$\frac{\partial^4 y}{\partial x^4} + \left(\frac{k}{EI}\right) y = 0$$

where E is the Young's modulus (N/m^2) and I is the moment of inertia (m^4) of the beam; also, k is the stiffness of the foundation (N/m2).

a) Derive the general solution of $y(x)$. You will find it convenient to define $\alpha = \sqrt[4]{\frac{k}{4EI}}$.

b) Find the solution of $y(x)$ for the special case of a point load of magnitude P applied downward at $x = 0$. Recall that the shear force, S, in the beam is given by $S = EIy'''$.

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MATHEMATICS

Problem 4

The gradient of a vector field $\mathbf{v}$ in a cylindrical coordinate system (r,θ,z) is given in matrix form by

$$[\operatorname{grad}\mathbf{v}]=\begin{bmatrix} \dfrac{\partial v_r}{\partial r} & \dfrac{1}{r}\left(\dfrac{\partial v_r}{\partial\theta}-v_\theta\right) & \dfrac{\partial v_r}{\partial z} \\ \dfrac{\partial v_\theta}{\partial r} & \dfrac{1}{r}\left(\dfrac{\partial v_\theta}{\partial\theta}+v_r\right) & \dfrac{\partial v_\theta}{\partial z} \\ \dfrac{\partial v_z}{\partial r} & \dfrac{1}{r}\dfrac{\partial v_z}{\partial\theta} & \dfrac{\partial v_z}{\partial z} \end{bmatrix},$$

where (v_r,v_θ,v_z) are the cylindrical components of $\mathbf{v}$.

a) Provide $\operatorname{div}\mathbf{v}$.
b) Provide $\operatorname{curl}\mathbf{v}$, knowing that it is equal to twice the dual vector of the antisymmetric part of $\operatorname{grad}\mathbf{v}$.

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MATHEMATICS

Problem 5

Find the general solution $u(x,t)$ to the equation

$$u_{tt} = c^2 u_{xx}, \qquad 0 \le x \le L,$$

subject to the boundary conditions

$$u(0,t) = 0, \qquad u(L,t) = 0,$$

and initial conditions

$$u(x,0) = f(x), \qquad u_t(x,0) = g(x),$$

where $f(x)$ and $g(x)$ are arbitrary functions.

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MATHEMATICS

Problem 6

Suppose a square matrix A has a repeated eigenvalue λ, and one cannot find more than one linearly independent eigenvector associated with that eigenvalue. Then people make use of generalized eigenvectors

$$(A-\lambda I)v_0 = 0$$
$$(A-\lambda I)v_1 = v_0$$
$$(A-\lambda I)v_2 = v_1$$
$$\vdots$$

The v_0 is the true eigenvector, v_1 is the first generalized eigenvector, v_2 is the second generalized eigenvector, ….

a) Suppose that A is a 3 by 3 matrix with all repeated eigenvalues and only one true eigenvector. Define matrix $M = \begin{bmatrix} v_0 & v_1 & v_2 \end{bmatrix}$ and show that $M^{-1}AM = J$ or $AM = MJ$ where J is the Jordan canonical form

$$J = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$$

b) If $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix}$ find M. Show that M^{-1} exists.

c) Give the definition of the exponential of a matrix e^{At} for any square matrix A, in terms of its power series.

d) Matrix J can be written as $J = \lambda I + \bar{J}$ where $\bar{J} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Find a closed form expression for the exponential of this Jordan form $e^{Jt} = e^{(\lambda I + \bar{J})t} = e^{\lambda It} e^{\bar{J}t}$ by summing the two series.

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MATHEMATICS

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Problem 1

For the function

$$\frac{1}{(z-1)(z-2)}$$

find the Laurent series expansion in

(A) the region $|z|<1$,

(B) the region $1<|z|<2$,

(C) the region $|z|>2$.

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MATHEMATICS

Problem 2

Given the following state variable differential equation

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -4 & -6 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Find the solution $\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ of this differential equation in terms of the initial condition $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$, by the following method: Find a transformation of state variables that diagonalizes the set of equations; solve the transformed differential equations; then convert the solution back to the original state variables.

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MATHEMATICS

Problem 3

(A) Consider an $m \times n$ matrix A, and an $n \times m$ matrix B. Make a careful proof that

$$tr(AB) = tr(B^T A^T)$$

by computing each of the indicated operations on the components of the matrices, including the appropriate limits on summations. The $tr(\bullet)$ represents the trace of the matrix in parentheses (i.e. the sum of the diagonal elements), and superscript T indicated the transpose.

(B) Using the same techniques, prove that the transpose of any product of two matrices is the product of the transposes in reverse order, i.e. show that

$$(CD)^T = D^T C^T$$

where C can be $m \times n$ and D can be $n \times k$.

(C) Given the result in (B), supply a different simple proof for (A).

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MATHEMATICS

Problem 4

Solve the partial differential equation

$$\frac{\partial^2 f}{\partial x^2} - \frac{\partial f}{\partial t} = 0, \quad 0 \le x \le 1, t \ge 0,$$

for $f(x,t)$, subject to the boundary conditions

$$\left.\frac{\partial f}{\partial x}\right|_{x=0} = q, \quad f(1,t) = 1,$$

where q is a constant, and the initial condition

$$f(x,0) = 0.$$

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MATHEMATICS

Problem 5

In a spherical coordinate system (r, θ, ϕ), let the basis vectors $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi\}$ be given by

$$\mathbf{e}_r = \mathbf{e}_1 \cos\theta \sin\phi + \mathbf{e}_2 \sin\theta \sin\phi + \mathbf{e}_3 \cos\phi$$
$$\mathbf{e}_\theta = -\mathbf{e}_1 \sin\theta + \mathbf{e}_2 \cos\theta$$
$$\mathbf{e}_\phi = \mathbf{e}_1 \cos\theta \cos\phi + \mathbf{e}_2 \sin\theta \cos\phi - \mathbf{e}_3 \sin\phi$$

where $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ represent a Cartesian basis. The position of a point in space is given by

$$\mathbf{x} = r\mathbf{e}_r$$

For any vector $\mathbf{v}(\mathbf{x}) = v_r \mathbf{e}_r + v_\theta \mathbf{e}_\theta + v_\phi \mathbf{e}_\phi$, determine its gradient $\operatorname{grad} \mathbf{v}$ in this spherical coordinate system.

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MATHEMATICS

Problem 6

Find the general solution to the following Ordinary Differential Equation

$$\frac{dy}{dx} - \frac{1}{x} y = \left(1 + \ln x\right) y^3$$

Hint: Try the transformation $v = y^{-2}$.

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MATHEMATICS

Problem 1

Solve

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=0, \qquad \begin{matrix} 0\le x\le 1 \\ 0\le y\le 1 \end{matrix}$$

for $T(x,y)$, subject to

$$T(0,y)=T_0, \qquad \left.\frac{\partial T}{\partial x}\right|_{x=1}+T(1,y)=0$$

$$\left.\frac{\partial T}{\partial y}\right|_{y=0}=0, \qquad \left.\frac{\partial T}{\partial y}\right|_{y=1}=0$$

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MATHEMATICS

Problem 2

A linear amplifier produces an output that is the convolution of its input and its response function. The Fourier transform of the response function of the amplifier is:

$$K(\omega)=\frac{i\omega}{\sqrt{2\pi}\,(\alpha+i\omega)^2},$$

where $\alpha>0$. Determine the output signal $g(t)$ if the input signal is the Heaviside step function, $h(t)$, defines as:

$$h(t)=\begin{cases} 0 \text{ if } t<0 \\ 1 \text{ if } t\geq 0 \end{cases}$$

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MATHEMATICS

Problem 3

Given the function $u(x,y) = e^{-x}\cos y + xy$:

(a) Show that $u(x,y)$ is harmonic over the complex domain.

(b) Find the family of curves $v(x,y)$ that are orthogonal to the family of curves $u(x,y) = c$ where c is a given constant. (*Hint:* use the Cauchy Riemann conditions for analytic functions).

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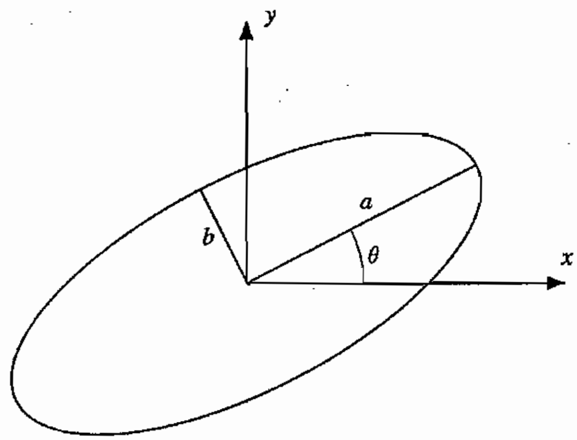
Problem 4



(a) What are the principal directions of this ellipse in the given coordinate system?
(b) What is the equation of this ellipse in the given coordinate system?
(c) What is the area moment of inertia matrix of this ellipse in the given coordinate system?
Hint: Along its principal directions, the area moment of inertia matrix of the ellipse is given by

$$I_p = \frac{\pi}{4} ab \begin{bmatrix} a^2 & 0 \\ 0 & b^2 \end{bmatrix}$$

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MATHEMATICS

Problem 5

(a) Consider the eigenvalue problem $y''+\lambda^2 y=0$ where λ is real. The boundary conditions are $y'(0)=0$ and $y(\ell)=0$. Determine the eigenvalues, λ_n, eigenfunctions, $y_n(x)$, and state the complete orthogonality condition. *Hint:* Consider all possible values of λ.

(b) For $\ell=1$, a function on the interval $0\le x\le 1$ that satisfies the boundary conditions is $f(x)=x^2(x-1)$. Expand this function in a Fourier series using the eigenfunctions determined in part (a), *i.e.* write $f(x)=\sum_{n=0}^{\infty}a_n y_n(x)$ and find the value of a_n for all appropriate values of the integer n.

Hint:

$$\int_0^1 x^3\cos(\lambda x)dx=\lambda^{-4}\left[\lambda^3\sin\lambda+3\lambda^2\cos\lambda-6\lambda\sin\lambda-6\cos\lambda+6\right]$$
$$\int_0^1 x^3\sin(\lambda x)dx=\lambda^{-4}\left[-\lambda^3\cos\lambda+3\lambda^2\sin\lambda+6\lambda\cos\lambda-6\sin\lambda\right]$$
$$\int_0^1 x^2\cos(\lambda x)dx=\lambda^{-3}\left[\lambda^2\sin\lambda+2\lambda\cos\lambda-2\sin\lambda\right]$$
$$\int_0^1 x^2\sin(\lambda x)dx=\lambda^{-3}\left[-\lambda^2\cos\lambda+2\lambda\sin\lambda+2\cos\lambda-2\right]$$

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MATHEMATICS

Problem 6

The singular value decomposition of a full rank matrix of dimension $n \times m$ where $n < m$ is given by

$$A = U\begin{bmatrix} S & 0 \end{bmatrix} V^T$$

where

S is a diagonal matrix, $S = diag[\sigma_1 \quad \sigma_2 \quad \cdots \quad \sigma_n]$
σ_i are the singular values, all real and greater than zero
U is an $n \times n$ matrix with the property that $U^{-1} = U^T$
V is an $m \times m$ matrix with the property that $V^{-1} = V^T$
and the 0 is a matrix of zeros of dimension $n \times (m - n)$

Consider the following system of equations

$$Ax = b$$

From the dimensions, there are more unknowns in column vector x than there are equations, so there are an infinite number of solutions.

(A) Show that the following $\hat{x}$ is a solution for x

$$\hat{x} = V\begin{bmatrix} S^{-1} \\ 0 \end{bmatrix} U^T b$$

(B) Show also that the following x^* is a solution for x

$$x^* = A^T (AA^T)^{-1} b$$

(C) Show that the columns of V are unit vectors, i.e. that the Euclidean norm (square root of the sum of the squares of the elements) is equal to unity.

(D) Define $\bar{x}$ such that $x = V \cdot \bar{x}$. Show that the Euclidean norm squared of x (i.e. $\|x\|^2 = x^T x$) and of $\bar{x}$ satisfy

$$\|\bar{x}\|^2 = \|x\|^2$$

(E) The equation $Ax = b$ uniquely determines some components of the $m \times 1$ vector $\bar{x}$. Write the set of all possible vectors $\bar{x}$ associated with solutions x of $Ax = b$. Then write the set of all possible solutions x to the original equation $Ax = b$.

(F) Demonstrate that the solution in (A) and the solution in (B) have the property that their Euclidean norm is the smallest of all possible solutions.

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MATHEMATICS

Problem 1:

a) Find the power series expansion for $f(z)=\dfrac{2}{(1+z)^2}$

b) Calculate the radius of convergence for this series using an appropriate test.

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MATHEMATICS

Problem 2:
For simplicity in this problem, consider vectors in three dimensional space written as a column matrix of the three components, e.g. $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

Conditions for the Norm of a Vector: People define various norms of such a vector, $\|x\|$. Any norm is required to satisfy three properties:

a) (a) $\|x\| = 0$ if and only if $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.
b) Let α be a scalar (either positive or negative). Then $\|\alpha x\| = |\alpha|\|x\|$ where αx multiplies each entry in the matrix by α.
c) The norm must satisfy the triangle inequality $\|x+y\| \le \|x\| + \|y\|$.

PART 1: Consider two candidate definitions for the norm, the infinity norm and the one norm

$$\|x\|_\infty = \max_i |x_i|$$

$$\|x\|_1 = |x_1| + |x_2| + |x_3|$$

and show that each satisfies the above three required conditions for a norm, (a), (b), and (c). ***Show*** all steps, and explain your reasoning. This is a mathematics question, so pay attention to being complete and rigorous, demonstrating "if and only if", etc.

PART 2: Conditions for the Norm of a Matrix: Consider the matrix equation $b = Ax$, or

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

People define the norm of matrix A that is induced by the definition of the norm of the vector, such that

$$\|b\|_\infty \le \|A\|_\infty \|x\|_\infty$$

$$\|b\|_1 \le \|A\|_1 \|x\|_1$$

By writing out the three scalar equations and taking absolute values, ***show*** that the induced infinity norm of matrix A can be written as

$$\|A\|_\infty = \max_i (|a_{i1}| + |a_{i2}| + |a_{i3}|)$$

Show every step in the logic, and be sure that each step is valid.

PART 3: ***Create*** an example of b, A, and x, with the property that equality holds, i.e.

$$\|b\|_\infty = \|A\|_\infty \|x\|_\infty$$

MATHEMATICS

Problem 3:
Derive Simpson's composite rule of integration to evaluate

$$I = \int_a^b f(x)dx$$

assuming that the function $f(x_i) = f_i$ is given at N uniformly spaced points $a \le x_i \le b$ ($i = 1 - N$). Clarify any constraints on N.

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MATHEMATICS

Problem 4:
Solve

$$y_{tt} = c^2 y_{xx} \qquad t > 0,\ 0 < x < L$$
$$y(0,t) = y(L,t) = 0$$
$$y(x,0) = f(x)$$
$$y_t(x,0) = 0$$

where

$$f(x) = \begin{cases} x & 0 < x < L/2 \\ L - x & L/2 < x < L \end{cases}$$

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MATHEMATICS

Problem 5:
Find the general solution to the ordinary differential equation

$$\frac{dy}{dx} = \frac{-3xy - 2y^2}{4x^2 + 5xy}.$$

Hint: Rearrange equation and multiply through by the integrating factor $x^m y^n$. Then find values of m and n to facilitate your solution.

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MATHEMATICS

Problem 6:
Use Fourier Tranforms to solve for $\phi(x,y)$:

$$\phi_{xx} = \phi_y + y\phi$$

for $-\infty < x < \infty$ and $y > 0$

subject to the boundary condition $\phi(x,0) = \delta(x)$, where $\delta(x)$ is the Dirac Delta function.

Use the following definitions for the Fourier transform, $F(\omega,y)$ of $\phi(x,y)$ and its inverse.

$$F(\omega,y) = \int_{-\infty}^{+\infty} \phi(x,y) e^{i\omega x}\,dx$$

$$\phi(x,y) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\omega,y) e^{-i\omega x}\,d\omega\,.$$

The following integral result may be useful

$$\int_{-\infty}^{+\infty} e^{-b\alpha^2}\,d\alpha = \sqrt{\frac{\pi}{b}}\,.$$

Problem 1

In a spherical coordinate system with coordinates (r,θ,ϕ), the orthonormal basis vectors $\{\mathbf{e}_r,\mathbf{e}_\theta,\mathbf{e}_\phi\}$ can be related to a Cartesian basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ through

$$\mathbf{e}_r = \cos\theta\sin\phi\,\mathbf{e}_1 + \sin\theta\sin\phi\,\mathbf{e}_2 + \cos\phi\,\mathbf{e}_3$$
$$\mathbf{e}_\theta = -\sin\theta\,\mathbf{e}_1 + \cos\theta\,\mathbf{e}_2$$
$$\mathbf{e}_\phi = \cos\theta\cos\phi\,\mathbf{e}_1 + \sin\theta\cos\phi\,\mathbf{e}_2 - \sin\phi\,\mathbf{e}_3$$

Derive the general expression for the components of the gradient of a scalar function $f(r,\theta,\phi)$ in spherical coordinates.

Problem 2

a) Solve the partial differential equation

$$\frac{\partial^2 c}{\partial z^2} - \frac{\partial c}{\partial t} = 0$$

for $c(z,t)$, subject to the boundary conditions

$$c(0,t) = 0\ ,\ c(1,t) = t$$

and initial condition

$$c(z,0) = 0$$

b) What is the behavior of the solution as $t \to \infty$?

MATHEMATICS

Doctoral Qualifying Examination, January 2008
Mechanical Engineering, Columbia University

Problem 3

Solve the following ODE:

Governing equation: $\dfrac{d^2\theta}{d\eta^2} = C\theta^{n+1} \qquad (C > 0)$

Boundary conditions: $\theta = 1$ at $\eta = 0$ and $\theta \to 0$ as $\eta \to \infty$

$\theta > 0$ for all $\eta \geq 0$

To solve the non-linear ODE, answer the following questions.

1. Using integration by parts, starting from $\int_{\eta}^{\infty} \theta \dfrac{d^2\theta}{d\zeta^2} d\zeta$, show that $d\theta/d\eta < 0$ for all $\eta \geq 0$.

2. Now that you know how θ and $d\theta/d\eta$ ($d\theta/d\eta$ also $\to 0$) behave as $\eta \to \infty$, solve the non-linear ODE. (Hint: Multiply both sides of the ODE by $d\theta/d\eta$ to convert the second order ODE to a first order ODE. Use the boundary conditions to determine unknown constants. If you know of any other method, feel free to solve it that way.)

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Problem 4

Consider the eigenvalue problem

$$x^2 y'' + xy' + \lambda y = 0$$

for $1 < x < L$ and with $y'(1) = 0$ and $y'(L) = 0$.

(a) Determine the eigenvalues and eigenfunctions of this equation.

(b) Given the eigenfunctions, e.g. call them $y_n(x)$, the eigenfunction expansion of an arbitrary piecewise smooth function $f(x)$ is

$$f(x) = \sum_{n=0}^{\infty} c_n y_n(x).$$

Determine the integral expression to be able to solve for the c_n. You are not expected to evaluate the integrals.

Hint: Write the differential equation in its Sturm-Liouville form and identify the weighting function.

Problem 5

Evaluate for real, positive values of a the following integral.

$$\int_{-\infty}^{+\infty} \frac{\sin^2 x}{a^2 + x^2} dx$$

Calculate the result as $a \to 0$.

Hint: Recall that $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$.

Problem 6

(A) The general solution of an nth order linear homogeneous ordinary differential equation is obtained by taking the sum of arbitrary constants times n linearly independent functions that satisfy the equation. Give a definition of what is meant by linearly independent functions.

You are making a mathematical definition, so write the definitions requested in this part and later parts of this problem the way they would be written in a mathematics textbook. It might help to make precise definitions to remember that all mathematical definitions are "if and only if", whether they actually use these words or not.

(B) Consider a set of m real valued vectors (column matrices) in n dimensional space $n \geq m$. Give a mathematical definition of what is meant when one says these vectors are linearly independent.

(C) For a set of m linearly independent vectors in n dimensional space, this time $n > m$, define what is meant by the space spanned by these vectors, and define what is meant by a set of basis functions for this space. Then create a method to produce an orthonormal basis for the space spanned by these vectors. Don't just apply a method that you have learned. Instead develop the method in such a way that when you are finished you have mathematically proved that the resulting vectors do form a spanning set, and do form an orthonormal basis – making use of your definitions of these words.

(D) Give a mathematical definition of the rank of an n by m matrix, where $n \geq m$. Describe various methods you could use to determine the rank of such a matrix. If you are using a computer to compute a matrix of interest and then to determine its rank, what method would you suggest to decide the rank?

(E) Given a real column matrix $b = [b_1 \quad b_2 \quad \cdots \quad b_n]^T$, the product bb^T produces a square matrix. What is the rank? Fully justify your answer mathematically.

MATHEMATICS

Doctoral Qualifying Examination, January 2007
Mechanical Engineering, Columbia University

Problem 1

a) Find the power series expansion $f(z)=\sum_{k=0}^{k=\infty} a_k z^k$ given the relation $f(z)(1+z)^2=2$

b) Calculate the radius of convergence for this series using an appropriate test.

MATHEMATICS

Doctoral Qualifying Examination, January 2007
Mechanical Engineering, Columbia University

Problem 2

The general solution to the ordinary differential equation $y'' - y = 0$ is $y = A\cosh(x) + B\sinh(x)$. Solve the differential equation using the Power Series method in order to determine the power series expansions about $x = 0$ for each of $\cosh(x)$ and $\sinh(x)$.

Problem 3

Solve the following partial differential equation for $c(x,t)$,

$$\frac{\partial^2 c}{\partial x^2} - \frac{\partial c}{\partial t} = 0\,,\ -\infty < x < \infty$$

subject to the boundary conditions

$$\lim_{x\to\pm\infty} c(x,t) = 0$$

and initial condition

$$c(x,0) = \delta(x)$$

where $\delta(x)$ is the Dirac delta function.

Hint: Recall that the Fourier integral transform is defined by

$$\mathcal{F}\{f(x)\} = \tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$$

Problem 4

Consider a circular annular region ($a \le r \le b$, $0 \le \theta \le 2\pi$) on which the Helmholtz equation

$$\nabla^2 u = \alpha^2 u$$

holds with boundary conditions $u(r = a, \theta) = 0$ and $u(r = b, \theta) = \sin^2\theta$, and with $\alpha > 0$. Find the solution $u(r, \theta)$.

Recall that in polar coordinates $\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$.

Problem 5

Consider the matrix equation

$$Ax = b$$

where A and b are given and we want to find the column vector x. Matrix A is of full rank, and is of dimension $m \times n$ where $m > n$. Therefore there are more scalar equations than unknowns in the components of x, and except under special circumstances, there will be no solution for x. One should write instead

$$Ax = b + \varepsilon$$

where ε is the column vector of the errors in satisfying each of the scalar component equations. Develop from basic principles the mathematics giving the value of x, call it $\hat{x}$, that will minimize the sum of the squares of these errors, i.e. minimize $\varepsilon^T \varepsilon$. Do not just use any memorized result, derive the result from basic principles.

Problem 6

Consider a matrix A whose singular value decomposition is given by

$$A = U\Sigma V^T$$

Suppose temporarily that matrix A is square.

(a) Show that the columns of matrix V are eigenvectors of $A^T A$.

(b) Show that the columns of matrix U are eigenvectors of AA^T.

(c) Show that the singular values are square roots of the eigenvalues of $A^T A$ and of AA^T.

Suppose matrix A is square, symmetric and diagonalizable.

(d) What is the relationship between the eigenvalues of matrix A, and the singular values of matrix A?

Now suppose that A is not square and of dimension $m \times n$ where $m > n$.

(e) What is modified in parts (a), (b), and (c) in this situation?

MATHEMATICS

Doctoral Qualifying Examination, January 2006
Mechanical Engineering, Columbia University

Problem 1

For the function

$$\frac{1}{(z-1)(z-2)}$$

find the Laurent series expansion in

(A) the region $|z|<1$
(B) the region $1<|z|<2$
(C) the region $|z|>2$

MATHEMATICS

Doctoral Qualifying Examination, January 2006
Mechanical Engineering, Columbia University

Problem 2
Solve

$$\left(\frac{x^2}{2}+xy\right)\frac{dy}{dx}=-\left(xy+\frac{y^2}{2}\right)$$

with the condition that $y(1)=1$.

Hint: Recall that if y depends upon x, then $\frac{d}{dx}\phi(x,y(x))=\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{\partial y}{\partial x}$

MATHEMATICS

Doctoral Qualifying Examination, January 2006
Mechanical Engineering, Columbia University

Problem 3

Solve the following partial differential equation for $c(r,t)$,

$$\frac{\partial^2 c}{\partial r^2}+\frac{1}{r}\frac{\partial c}{\partial r}-\frac{1}{D}\frac{\partial c}{\partial t}=0\,,\ 0\le r\le a$$

subject to the boundary conditions

$$\left.\frac{\partial c}{\partial r}\right|_{r=0}=0\,,\ c(a,t)=c^*$$

and initial condition:

$$c(r,0)=0\ .$$

Hint: $J_0(0)=1$, $J_1(0)=0$, $J_0'(x)=-J_1(x)$, $\int_0^b xJ_0(x)=bJ_1(b)$,

$$\int_0^b xJ_0(\alpha x)J_0(\beta x)dx=\begin{cases}\dfrac{b}{\alpha^2-\beta^2}\left[\alpha J_0(\beta b)J_1(\alpha b)-\beta J_0(\alpha b)J_1(\beta b)\right] & \alpha\ne\beta\\[2ex] \dfrac{b^2}{2}\left[J_0^2(\alpha b)+J_1^2(\alpha b)\right] & \alpha=\beta\end{cases}.$$

MATHEMATICS

Doctoral Qualifying Examination, January 2006
Mechanical Engineering, Columbia University

Problem 4

The parametric representation of a surface in 3D space is given by

$$\mathbf{x}(\theta,\varphi)=x_1(\theta,\varphi)\mathbf{e}_1+x_2(\theta,\varphi)\mathbf{e}_2+x_3(\theta,\varphi)\mathbf{e}_3$$

where the vector $\mathbf{x}$ represents the position of a point on the surface, $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ are Cartesian basis vectors and

$$x_1(\theta,\varphi)=a\cos\theta\sin\varphi$$
$$x_2(\theta,\varphi)=b\sin\theta\sin\varphi \quad \text{with } 0\le\theta\le 2\pi\,,\ 0\le\varphi\le\pi$$
$$x_3(\theta,\varphi)=c\cos\varphi$$

(A) Find the algebraic representation of this surface. What is its shape?

(B) Find the unit outward normal to this surface.

(C) Find the intersection of this surface with the plane $(\mathbf{x}-\mathbf{p})\cdot\mathbf{n}=0$, where $\mathbf{p}$ is a fixed point in space which lies in the plane and $\mathbf{n}$ is a unit vector normal to the plane. For the special case where $\mathbf{n}=\mathbf{e}_3$, determine under which condition the solution exists, show that this curve is an ellipse, and find its major and minor radii.

MATHEMATICS

Doctoral Qualifying Examination, January 2006
Mechanical Engineering, Columbia University

Problem 5

(A) Find the solution $\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ of the following equation satisfying $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}$ initial conditions

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ -4 & -5 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

(B) Find a particular solution of the following differential equation

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ -4 & -5 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(C) Find the solution to the equation in (B) satisfying the initial conditions in (A).

MATHEMATICS

Doctoral Qualifying Examination, January 2006
Mechanical Engineering, Columbia University

Problem 6

(A) Prove that the eigenvalues of a matrix are unaffected by a similarity transformation, i.e. the eigenvalues of $M^{-1}AM$ are the same as those of matrix A. How are the eigenvectors for each related?

(B) Suppose that A is real and symmetric. If λ_i and λ_j are two distinct eigenvalues of A, show that the associated eigenvectors v_i and v_j are orthogonal, i.e. show that $v_i^T v_j = 0$.

(C) For the same A as in part (B), if all eignevalues are distinct, show that A can be diagonalized by a similarity transformation $M^{-1}AM$ with $M^{-1} = M^T$.

(D) Two definitions of a positive definite matrix A (real and symmetric) are

(i) A is positive definite if and only if $x^T Ax > 0$ for all vectors x that are not identically zero.
(ii) A is positive definite if and only if all eigenvalues of A are greater than zero.

Show that these two definitions are equivalent (assuming all eigenvalues are distinct). Do a careful development, stating in detail the logic you use, and explicitly stating any properties of symmetric matrices that you need.

Note: $\det(AB) = \det(A)\det(B)$ provided all determinants exist.

MATHEMATICS **Doctoral Qualifying Examination, January 2005**
Mechanical Engineering, Columbia University

Problem 1

Let $z = x + iy \in C$, $i \equiv \sqrt{-1}$. Calculate the following integrals along the path shown in figure 1.

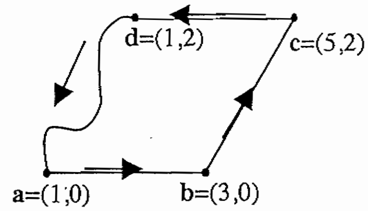

Figure 1

a) Calculate the integral $\int_a^b (iz+5)dz$ along the path in Figure 1.

b) Calculate the integral $\int_b^c (iz+5)dz$ along the path in Figure 1.

c) Find the value of $\int_c^a (iz+5)dz$ without explicit calculation of the integral. Support your answer with a mathematical proof. without explicit calculation of the integral.

Note: arrows indicate the directions along which the path is traversed during integration.

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MATHEMATICS

Doctoral Qualifying Examination, January 2005
Mechanical Engineering, Columbia University

Problem 2

Calculate the minimal and maximal height (where height is defined by the z-coordinate) along the conic defined by the intersection of $z^2 - x^2 - y^2 = 0$ with $5x + 3y + z - 5 = 0$. Explain the geometric meaning of this problem.

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MATHEMATICS **Doctoral Qualifying Examination, January 2005**
Mechanical Engineering, Columbia University

Problem 3

Given the matrix

$$[A]=\begin{bmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 3\end{bmatrix}$$

a) What are the eigenvalues and eigenvectors of A?
b) Evaluate A^{-1}
c) What are the invariants of A?
d) Show that A satisfies its own characteristic equation.
e) Given the matrix

$$[Q]=\begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\ 0 & 0 & 1\end{bmatrix}$$

show that Q is an orthogonal transformation.

f) What are the eigenvalues and eigenvectors of $[B]=[Q]^T[A][Q]$?

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MATHEMATICS

Doctoral Qualifying Examination, January 2005
Mechanical Engineering, Columbia University

Problem 4

Derive Simpson's rule for numerical integration of a function $f(x)$,

$$I = \int_a^b f(x)dx$$

using n uniform increments Δx over $a \leq x \leq b$.

MATHEMATICS **Doctoral Qualifying Examination, January 2005**
Mechanical Engineering, Columbia University

Problem 5

Solve the following variant of the Laplace equation defined as:

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial\phi}{\partial x}=0$$

With boundary conditions

$$\frac{\partial\phi}{\partial y}(x,y=0)=0\ ,\ \phi(x,y=1)=0\ ,\ \frac{\partial\phi}{\partial x}(x=0,y)=-1\ ,\ \phi(x\to\infty,y)\to 0\ .$$

Hint: Use the technique of separation of variables in which $\phi(x,y)=X(x)Y(y)$.

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Problem 6

Consider the following ODE:

$$\frac{d^2y}{dx^2} - e^{4x}y = 0$$

Find the general solution to the equations in terms of well-known functions.

Hint: Make the change of variable $z = e^{cx}$ and choose c so that you have an equation whose solution you know and then write down the two linearly independent solutions.

Problem 1

The equation

$$\frac{d^2y}{dx^2} + x^4 y = 0$$

has solutions of the form $y(x) = x^{\frac{1}{2}} f(x^3/3)$. Make this change of variable and determine the general solution to this second-order differential equation. To receive credit, you must clearly illustrate the change of variables and subsequent steps.

Problem 2

Solve the following partial differential equation for $\varphi(x,y)$

$$\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}=-2c,\ -a\le x\le a,\ -b\le y\le b$$

subject to the boundary conditions

$$\varphi=0 \text{ on } x=\pm a \text{ and } y=\pm b$$

Problem 3

Find the solution for the following initial value problem:

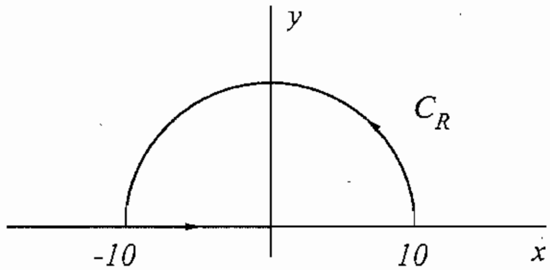
$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

$$u(x,0) = 0,\ u(0,t) = 0,\ u(1,t) = t\,.$$

Problem 4

(a) Find the roots for $z^4 + 1 = 0$

(b) Evaluate the integral $\oint_{C_R} \frac{dz}{z^4+1}$ where C_R is as shown:



Problem 5

(A) Consider an n by n matrix A having distinct eigenvalues. Show that it can be written as

$$A = A_1 + A_2$$

where

$$A_1 = \tfrac{1}{2}(A + A^T) \qquad A_2 = \tfrac{1}{2}(A - A^T)$$

(B) Define the terms symmetric and anti symmetric matrix, and show that A_1 is a symmetric matrix and that A_2 is anti symmetric.

(C) Show that $x^T Ax = x^T A_1 x$.

(D) Let λ_i be the i^{th} eigenvalue and v_i the associated eigenvector of A_1. Write the standard eigenvalue equation for this matrix. Show that the length of each v_i (the square root of the sum of the squares of the components) can be made any nonzero number you might want.

(F) Prove that

$$v_i^T v_j = 0 \text{ for } i \neq j$$

(G) Let M be the matrix of eigenvectors, $M = \begin{bmatrix} v_1 & v_2 & \ldots & v_n \end{bmatrix}$, each adjusted to length one. Show that $M^T M = I$ (the identity matrix) and $M^{-1} = M^T$.

(H) Show that $A_1 M = M\Lambda$ where M is the diagonal matrix of eigenvalues, $\Lambda = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$.

(I) Prove that $x^T Ax > 0$ for all x not identically zero, if and only if all $\lambda_i > 0$.