727

FLUID MECHANICS
Doctoral Qualification Exam, May 2016
Mechanical Engineering Department, Columbia University

Problem 1

For a flow field given by

$$\vec{\mathbf{v}}(\vec{x},t) = \frac{x_1}{1+t}\mathbf{e}_1 + \frac{x_2}{2+t}\mathbf{e}_2,$$

where $\mathbf{e}_1$ and $\mathbf{e}_2$ are unit vectors along the x_1 and x_2 axes respectively, find the equations for (1) the streamlines at t passing through the point $(x_{1,0}, x_{2,0})$, and (2) the pathline for a particle passing through the location $(2,3)$ at $t=0$.

Problem 2

To identify a point in a continuum, we specify its coordinates in a given reference frame at $t=0$ as (X_1, X_2, X_3). Let the coordinates of the same point at $t>0$ be given by $x_i = \chi_i(X_1, X_2, X_3, t)$, $(i = 1,2,3)$. In vector notation, the transformation can be written as $\vec{x} = \boldsymbol{\chi}(\vec{A}, t)$, and the inverse transformation as $\vec{A} = \boldsymbol{\chi}^{-1}(\vec{x}, t)$. The ratio of a volume occupied by an infinitesimal material region at time t to its volume at $t=0$ is defined as the Jacobian of the transformation.

1. Show that for the transformation $x_i = \chi_i(X_1, X_2, X_3, t)$, $(i = 1,2,3)$

$$J(X_1,X_2,X_3,t) = \begin{vmatrix} \frac{\partial\chi_1}{\partial X_1} & \frac{\partial\chi_1}{\partial X_2} & \frac{\partial\chi_1}{\partial X_3} \\ \frac{\partial\chi_2}{\partial X_1} & \frac{\partial\chi_2}{\partial X_2} & \frac{\partial\chi_2}{\partial X_3} \\ \frac{\partial\chi_3}{\partial X_1} & \frac{\partial\chi_3}{\partial X_2} & \frac{\partial\chi_3}{\partial X_3} \end{vmatrix} \quad \text{or} \quad \begin{vmatrix} \frac{\partial\chi_1}{\partial X_1} & \frac{\partial\chi_2}{\partial X_1} & \frac{\partial\chi_3}{\partial X_1} \\ \frac{\partial\chi_1}{\partial X_2} & \frac{\partial\chi_2}{\partial X_2} & \frac{\partial\chi_3}{\partial X_2} \\ \frac{\partial\chi_1}{\partial X_3} & \frac{\partial\chi_2}{\partial X_3} & \frac{\partial\chi_3}{\partial X_3} \end{vmatrix}$$

2. Find the Jacobian for the transformation $x_1 = X_1 e^t, x_2 = X_2 e^{-t}, x_3 = X_3$ and show that $\frac{DJ}{Dt} = J(\nabla\cdot\vec{\mathbf{v}})$, where $\frac{D}{Dt}$ is the substantial or material derivative, $\vec{\mathbf{v}} = \vec{\mathbf{v}}(\vec{x},t)$ is the velocity in spatial coordinates, and $\nabla\cdot\vec{\mathbf{v}} = \text{div }\vec{\mathbf{v}}$ is the divergence of $\vec{\mathbf{v}}$ in spatial coordinates.

3. Given that $\frac{DJ}{Dt} = J(X_1,X_2,X_3,t)(\nabla\cdot\vec{\mathrm{v}})$, show that

$$\frac{d}{dt}\int_{V(t)} \rho\phi dV = \int_{V(t)} \rho\frac{D\phi}{Dt}dV,$$

where $\phi(\vec{x},t)$ is any scalar property of the fluid, and $\rho(\boldsymbol{x},t)$ is the density of the fluid.

Problem 3 on next page

FLUID MECHANICS
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Problem 3

Consider a stationary bubble of incondensible gas in a infinite domain of incompressible liquid at uniform temperature. At $t = 0$, the pressure in the liquid is $p_\infty(0)$ and the radius of the bubble is R_0. The pressure in the liquid reduces to $p_\infty(t) = p_\infty^*$ for $t > 0$, resulting in an expansion of the bubble. The expansion is a spherically symmetric process. Answer the following questions regarding the expansion process:

1. Since the liquid is incompressible, show that the radial velocity $u(r,t) = \dfrac{F(t)}{r^2}$ for $r \geq R(t)$. From the kinematic boundary condition at $r = R(t)$, find the relation between $F(t)$ and $R(t), \dot{R}(t)$.

2. The density and viscosity of the liquid are ρ_L and μ_L respectively. Substitute the expression $u(r,t) = \dfrac{F(t)}{r^2}$ into the Navier-Stokes equation (see appendix for Navier-Stokes equation in spherical coordinates) to obtain the following equation:

$$-\frac{1}{\rho_L}\frac{\partial p}{\partial r} = \frac{1}{r^2}\dot{F}(t) - \frac{2F^2}{r^5},$$

where $p(r,t)$ is the pressure at time t and $r \geq R(t)$.

3. Integrate the above equation to show that $\dfrac{p(r,t) - p_\infty(t)}{\rho_L} = \dfrac{\dot{F}}{r} - \dfrac{F^2}{2r^4}$ and $\dfrac{p(R,t) - p_\infty(t)}{\rho_L} = R\ddot{R} + \dfrac{3}{2}\dot{R}^2$.

4. If the gas-liquid interface has a surface tension σ, the pressure inside and outside a stationary bubble are related by $p_{in} - p_{out} = \dfrac{2\sigma}{R}$. When the bubble is expanding, p_{out} has to be replaced by the appropriate component of the stress tensor. Using the expressions for the components of the stress tensor given in the appendix, show that $p_B(t) - \left[p(R,t) + 4\mu_L\dfrac{\dot{R}}{R}\right] = \dfrac{2\sigma}{R}$.

5. Assuming that the behavior of the gas in the bubble is polytropic, i.e, $p_B(0)R_0^{3k} = p_B(t)R^{3k}$ $(k > 1)$, show that $p_B(0)\dfrac{R_0^{3k}}{R^{3k}} - p_\infty(t) = \dfrac{2\sigma}{R} + 4\mu_L\dfrac{\dot{R}}{R} + \rho_L\left(R\ddot{R} + \dfrac{3}{2}\dot{R}^2\right)$.

6. Under what conditions can the viscous term in the above expression be neglected?

Problem 4 on next page

7α9

FLUID MECHANICS
Doctoral Qualification Exam, May 2016
Mechanical Engineering Department, Columbia University

Problem 4
Inviscid fluid at velocity U_0 flows around a cylinder of radius b located at the origin and rotating with an angular velocity ω (see Fig. 1). Answer the following questions:

1. Use superposition to construct the complex potential $F(z)$ of uniform flow around a rotating cylinder. **Hint**: What is the complex potential of uniform flow? How is it modified because of the presence of a stationary cylinder? What additional term is necessary when the cylinder is rotating? Commonly used complex potentials are Az, $B \ln z$, $\frac{C}{z}$, where A, B, C are appropriately chosen (complex) constants.
2. Find the velocity and pressure distribution on the boundary of the cylinder.
3. Determine the lift and drag force on the cylinder.

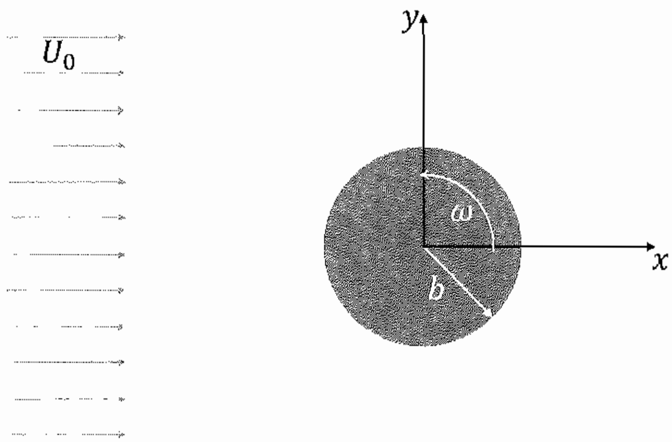


Figure 1: Problem 4

730

B3. Spherical Polar Coordinates

The spherical polar coordinates used are (r, θ, φ), where φ is the azimuthal angle (Figure 3.1c). Equations are presented assuming ψ is a scalar, and

$$\mathbf{u} = \mathbf{i}_r u_r + \mathbf{i}_\theta u_\theta + \mathbf{i}_\varphi u_\varphi,$$

where $\mathbf{i}_r$, $\mathbf{i}_\theta$, and $\mathbf{i}_\varphi$ are the local unit vectors at a point.

7.31

Gradient of a scalar

$$\nabla\psi = \mathbf{i}_r \frac{\partial\psi}{\partial r} + \mathbf{i}_\theta \frac{1}{r}\frac{\partial\psi}{\partial\theta} + \mathbf{i}_\varphi \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\varphi}.$$

Laplacian of a scalar

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\varphi^2}.$$

Divergence of a vector

$$\nabla\cdot\mathbf{u} = \frac{1}{r^2}\frac{\partial(r^2u_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(u_\theta\sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial\varphi}.$$

Curl of a vector

$$\nabla\times\mathbf{u} = \frac{\mathbf{i}_r}{r\sin\theta}\left[\frac{\partial(u_\varphi\sin\theta)}{\partial\theta} - \frac{\partial u_\theta}{\partial\varphi}\right] + \frac{\mathbf{i}_\theta}{r}\left[\frac{1}{\sin\theta}\frac{\partial u_r}{\partial\varphi} - \frac{\partial(ru_\varphi)}{\partial r}\right] + \frac{\mathbf{i}_\varphi}{r}\left[\frac{\partial(ru_\theta)}{\partial r} - \frac{\partial u_r}{\partial\theta}\right].$$

Laplacian of a vector

$$\begin{aligned}\nabla^2\mathbf{u} = \mathbf{i}_r&\left[\nabla^2u_r - \frac{2u_r}{r^2} - \frac{2}{r^2\sin\theta}\frac{\partial(u_\theta\sin\theta)}{\partial\theta} - \frac{2}{r^2\sin\theta}\frac{\partial u_\varphi}{\partial\varphi}\right]\\ &+ \mathbf{i}_\theta\left[\nabla^2u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial\theta} - \frac{u_\theta}{r^2\sin^2\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\varphi}{\partial\varphi}\right]\\ &+ \mathbf{i}_\varphi\left[\nabla^2u_\varphi + \frac{2}{r^2\sin\theta}\frac{\partial u_r}{\partial\varphi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\theta}{\partial\varphi} - \frac{u_\varphi}{r^2\sin^2\theta}\right].\end{aligned}$$

Strain rate and viscous stress (for incompressible form $\sigma_{ij} = 2\mu e_{ij}$)

$$\begin{aligned}e_{rr} &= \frac{\partial u_r}{\partial r} = \frac{1}{2\mu}\sigma_{rr},\\ e_{\theta\theta} &= \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_r}{r} = \frac{1}{2\mu}\sigma_{\theta\theta},\\ e_{\varphi\varphi} &= \frac{1}{r\sin\theta}\frac{\partial u_\varphi}{\partial\varphi} + \frac{u_r}{r} + \frac{u_\theta\cot\theta}{r} = \frac{1}{2\mu}\sigma_{\varphi\varphi},\\ e_{\theta\varphi} &= \frac{\sin\theta}{2r}\frac{\partial}{\partial\theta}\left(\frac{u_\varphi}{\sin\theta}\right) + \frac{1}{2r\sin\theta}\frac{\partial u_\theta}{\partial\varphi} = \frac{1}{2\mu}\sigma_{\theta\varphi},\\ e_{\varphi r} &= \frac{1}{2r\sin\theta}\frac{\partial u_r}{\partial\varphi} + \frac{r}{2}\frac{\partial}{\partial r}\left(\frac{u_\varphi}{r}\right) = \frac{1}{2\mu}\sigma_{\varphi r},\\ e_{r\theta} &= \frac{r}{2}\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{2r}\frac{\partial u_r}{\partial\theta} = \frac{1}{2\mu}\sigma_{r\theta}.\end{aligned}$$

732

Vorticity

$$\omega_r = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(u_\varphi \sin\theta) - \frac{\partial u_\theta}{\partial\varphi}\right],$$

$$\omega_\theta = \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial u_r}{\partial\varphi} - \frac{\partial(ru_\varphi)}{\partial r}\right],$$

$$\omega_\varphi = \frac{1}{r}\left[\frac{\partial}{\partial r}(ru_\theta) - \frac{\partial u_r}{\partial\theta}\right].$$

Equation of continuity

$$\frac{\partial\rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(\rho r^2 u_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\rho u_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}(\rho u_\varphi) = 0.$$

Navier–Stokes equations with constant ρ and ν, and no body force

$$\frac{\partial u_r}{\partial t} + (\mathbf{u}\cdot\nabla)u_r - \frac{u_\theta^2 + u_\varphi^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left[\nabla^2 u_r - \frac{2u_r}{r^2} - \frac{2}{r^2\sin\theta}\frac{\partial(u_\theta\sin\theta)}{\partial\theta} - \frac{2}{r^2\sin\theta}\frac{\partial u_\varphi}{\partial\varphi}\right],$$

$$\frac{\partial u_\theta}{\partial t} + (\mathbf{u}\cdot\nabla)u_\theta + \frac{u_r u_\theta}{r} - \frac{u_\varphi^2\cot\theta}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial\theta} + \nu\left[\nabla^2 u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial\theta} - \frac{u_\theta}{r^2\sin^2\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\varphi}{\partial\varphi}\right],$$

$$\frac{\partial u_\varphi}{\partial t} + (\mathbf{u}\cdot\nabla)u_\varphi + \frac{u_\varphi u_r}{r} + \frac{u_\theta u_\varphi\cot\theta}{r} = -\frac{1}{\rho r\sin\theta}\frac{\partial p}{\partial\varphi} + \nu\left[\nabla^2 u_\varphi + \frac{2}{r^2\sin\theta}\frac{\partial u_r}{\partial\varphi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\theta}{\partial\varphi} - \frac{u_\varphi}{r^2\sin^2\theta}\right],$$

where

$$\mathbf{u}\cdot\nabla = u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial\theta} + \frac{u_\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi},$$

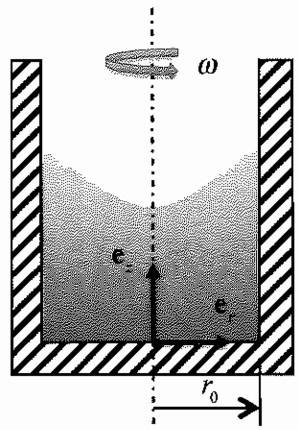
$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}.$$

674

FLUIDS
Doctoral Qualifying Examination, January 2016
Mechanical Engineering Department, Columbia University

Problem 1

Hints on last page.



An incompressible liquid is placed inside a cylindrical container with inner radius r_0, and spun about the cylinder axis at a constant angular speed ω. At steady state, the velocity of the liquid is given by $\mathbf{v} = \omega r\,\mathbf{e}_\theta$ and its pressure is $p = \rho\left(\frac{1}{2}\omega^2 r^2 - gz\right) + p_0$ where ρ is the liquid's density, g is the gravitational acceleration (i.e., the body force is $\mathbf{b} = -g\mathbf{e}_z$), and p_0 is the fluid pressure at the bottom center of the cylinder ($r = 0$, $z = 0$).

a) Verify that the given fluid velocity satisfies the axiom of mass balance.
b) Evaluate the acceleration $\mathbf{a}$ of the fluid.
c) Evaluate $\operatorname{curl}\mathbf{v}$.
d) For an incompressible Newtonian fluid, the stress tensor is given by $\mathbf{T} = -p\mathbf{I} + 2\mu\mathbf{D}$, where $\mathbf{I}$ is the identity tensor, $\mathbf{D}$ is the symmetric part of $\operatorname{grad}\mathbf{v}$, and μ is the viscosity. For the given velocity and pressure, evaluate the stress $\mathbf{T}$.
e) Verify that your solution for $\mathbf{T}$ satisfies the axiom of linear momentum balance.
f) Evaluate the traction vector on the bottom face of the container.
g) Evaluate the traction on the lateral face of the container.
h) The free surface of the liquid must be traction-free. What is the profile (shape) of the free surface?
i) Given p_0, what is the initial height of the liquid surface (before spinning)?

FLUIDS
Doctoral Qualifying Examination, January 2016
Mechanical Engineering Department, Columbia University

Problem 2

a) Prove that $(\mathrm{grad}\,\mathbf{v})\cdot\mathbf{v} = \boldsymbol{\omega}\times\mathbf{v} + \mathrm{grad}\left(\frac{1}{2}\mathbf{v}\cdot\mathbf{v}\right)$, where $\boldsymbol{\omega} = \mathrm{curl}\,\mathbf{v}$.

b) Derive Bernoulli's equation from the linear momentum balance equation, assuming that the flow is incompressible, steady, and inviscid, and that the external body forces are conservative.

FLUIDS
Doctoral Qualifying Examination, January 2016
Mechanical Engineering Department, Columbia University

Problem 3

Consider a semi-infinite fluid domain above an infinite plate located in the $x_1 - x_3$ plane, in the absence of body forces and external pressure gradients. The plate oscillates sinusoidally along x_1 with an amplitude v_0 and angular frequency ω. The fluid is incompressible and Newtonian. Solve for the fluid velocity above the plate.

FLUIDS
Doctoral Qualifying Examination, January 2016
Mechanical Engineering Department, Columbia University

Problem 4

The axiom of entropy inequality is given by

$$-\rho\left(\dot{\psi}+\eta\dot{\theta}\right)+\mathbf{T}:\mathbf{D}-\frac{1}{\theta}\mathbf{q}\cdot\operatorname{grad}\theta\geq 0$$

where ψ is the specific free energy, η is the specific entropy, $\mathbf{T}$ is the Cauchy stress, $\mathbf{q}$ is the heat flux, ρ is the density, θ is the temperature and $\mathbf{D}=\left(\operatorname{grad}\mathbf{v}+\operatorname{grad}^T\mathbf{v}\right)/2$ is the rate of deformation.

a) Show that $\dot{J}=J\operatorname{div}\mathbf{v}$, where $J=\det\mathbf{F}$ and $\mathbf{F}$ is the deformation gradient.

b) To model viscous compressible fluids under isothermal conditions, we restrict our choice of state variables to θ, J and $\mathbf{D}$. What constraints does the entropy inequality now impose on the functions of state ψ, η, $\mathbf{T}$ and $\mathbf{q}$?

c) For a compressible Newtonian viscous fluid, show that the entropy inequality implies that the bulk and shear viscosities must be positive.

FLUIDS
Doctoral Qualifying Examination, January 2016
Mechanical Engineering Department, Columbia University

Hints

Cylindrical coordinates

Gradient of a scalar field ϕ

$$\operatorname{grad}\phi = \frac{\partial \phi}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial \phi}{\partial \theta}\mathbf{e}_\theta + \frac{\partial \phi}{\partial z}\mathbf{e}_z$$

Gradient of a vector field $\mathbf{v}$

$$[\operatorname{grad}\mathbf{v}] = \begin{bmatrix} \dfrac{\partial v_r}{\partial r} & \dfrac{1}{r}\left(\dfrac{\partial v_r}{\partial \theta} - v_\theta\right) & \dfrac{\partial v_r}{\partial z} \\ \dfrac{\partial v_\theta}{\partial r} & \dfrac{1}{r}\left(\dfrac{\partial v_\theta}{\partial \theta} + v_r\right) & \dfrac{\partial v_\theta}{\partial z} \\ \dfrac{\partial v_z}{\partial r} & \dfrac{1}{r}\dfrac{\partial v_z}{\partial \theta} & \dfrac{\partial v_z}{\partial z} \end{bmatrix}$$

Divergence of a vector field $\mathbf{v}$

$$\operatorname{div}\mathbf{v} = \frac{\partial v_r}{\partial r} + \frac{1}{r}\left(\frac{\partial v_\theta}{\partial \theta} + v_r\right) + \frac{\partial v_z}{\partial z}$$

Laplacian of a vector field $\mathbf{v}$

$$\begin{aligned} \operatorname{lap}\mathbf{v} = & \left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial (r v_r)}{\partial r}\right) + \frac{1}{r^2}\left(\frac{\partial^2 v_r}{\partial \theta^2} - 2\frac{\partial v_\theta}{\partial \theta}\right) + \frac{\partial^2 v_r}{\partial z^2}\right)\mathbf{e}_r \\ & + \left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial (r v_\theta)}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{\partial^2 v_\theta}{\partial z^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta}\right)\mathbf{e}_\theta \\ & + \left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right)\mathbf{e}_z \end{aligned}$$

Divergence of a tensor field $\mathbf{T}$

$$\begin{aligned} \operatorname{div}\mathbf{T} = & \left(\frac{\partial T_{rr}}{\partial r} + \frac{1}{r}\left(\frac{\partial T_{r\theta}}{\partial \theta} + T_{rr} - T_{\theta\theta}\right) + \frac{\partial T_{rz}}{\partial z}\right)\mathbf{e}_r \\ & + \left(\frac{\partial T_{\theta r}}{\partial r} + \frac{1}{r}\left(\frac{\partial T_{\theta\theta}}{\partial \theta} + T_{\theta r} + T_{r\theta}\right) + \frac{\partial T_{\theta z}}{\partial z}\right)\mathbf{e}_\theta \\ & + \left(\frac{\partial T_{zr}}{\partial r} + \frac{1}{r}\left(\frac{\partial T_{z\theta}}{\partial \theta} + T_{zr}\right) + \frac{\partial T_{zz}}{\partial z}\right)\mathbf{e}_z \end{aligned}$$

Doctoral Qualifying Examination, January 2015
Mechanical Engineering Department, Columbia University
FLUID MECHANICS

Problem 1

An incompressible Newtonian fluid maintains a steady flow under the action of a prescribed uniform pressure gradient $\partial p/\partial z$ along a horizontal annular pipe, in the absence of body forces. The inner and outer radii of the pipe are R_i and R_o, respectively.

a) Starting from the continuity and Navier-Stokes equations in cylindrical coordinates, write down the governing system of equations and the appropriate boundary conditions, carefully outlining your assumptions.
b) Calculate the velocity field.
c) Calculate the volumetric flow rate Q in the pipe.

Hint: Differential operators in cylindrical coordinates

$$\nabla\cdot\mathbf{v}=\frac{\partial v_r}{\partial r}+\frac{1}{r}\left(\frac{\partial v_\theta}{\partial \theta}+v_r\right)+\frac{\partial v_z}{\partial z}$$

$$\left(\nabla\varphi\right)_r=\frac{\partial\varphi}{\partial r},\quad \left(\nabla\varphi\right)_\theta=\frac{1}{r}\frac{\partial\varphi}{\partial\theta},\quad \left(\nabla\varphi\right)_z=\frac{\partial\varphi}{\partial z}$$

$$\left(\nabla^2\mathbf{v}\right)_r=\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(rv_r\right)\right)+\frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{\partial v_r}{\partial\theta}-2v_\theta\right)+\frac{\partial^2 v_r}{\partial z^2}$$

$$\left(\nabla^2\mathbf{v}\right)_\theta=\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(rv_\theta\right)\right)+\frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{\partial v_\theta}{\partial\theta}+2v_r\right)+\frac{\partial^2 v_\theta}{\partial z^2}$$

$$\left(\nabla^2\mathbf{v}\right)_z=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_z}{\partial\theta^2}+\frac{\partial^2 v_z}{\partial z^2}$$

Navier-Stokes equation

$$\rho\frac{D\mathbf{v}}{Dt}=-\nabla p+\left(\lambda+\mu\right)\nabla\left(\nabla\cdot\mathbf{v}\right)+\mu\nabla^2\mathbf{v}+\rho\mathbf{b}$$

Doctoral Qualifying Examination, January 2015
Mechanical Engineering Department, Columbia University
FLUID MECHANICS

Problem 2

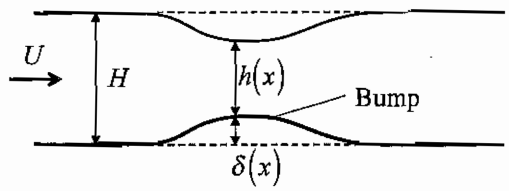
The spatial description of a continuum flow is given by $\mathbf{x} = \phi(\xi, t)$ where the components of $\mathbf{x}$ are

$$x_1 = \frac{1}{2}(\xi_1 + \xi_2)e^t + \frac{1}{2}(\xi_1 - \xi_2)e^{-t}$$
$$x_2 = \frac{1}{2}(\xi_1 + \xi_2)e^t - \frac{1}{2}(\xi_1 - \xi_2)e^{-t}$$
$$x_3 = \xi_3$$

a) Evaluate the Jacobian for this motion.
b) Is the flow incompressible? Does a material volume expand or shrink in time?
c) Determine the material description $\xi = \phi^{-1}(\mathbf{x}, t)$.
d) Calculate the velocity and acceleration fields in terms of the material coordinates.
e) Calculate the velocity and acceleration fields in terms of the spatial coordinates.
f) Calculate the velocity gradient tensor $\mathbf{L}$.
g) Calculate the rate of deformation tensor $\mathbf{D}$.
h) Assuming a Newtonian fluid, calculate the stress tensor $\mathbf{T} = -p\mathbf{I} + \lambda(\mathrm{tr}\,\mathbf{D})\mathbf{I} + 2\mu\mathbf{D}$.
i) Calculate the three invariants of the stress tensor.

Doctoral Qualifying Examination, January 2015
Mechanical Engineering Department, Columbia University
FLUID MECHANICS

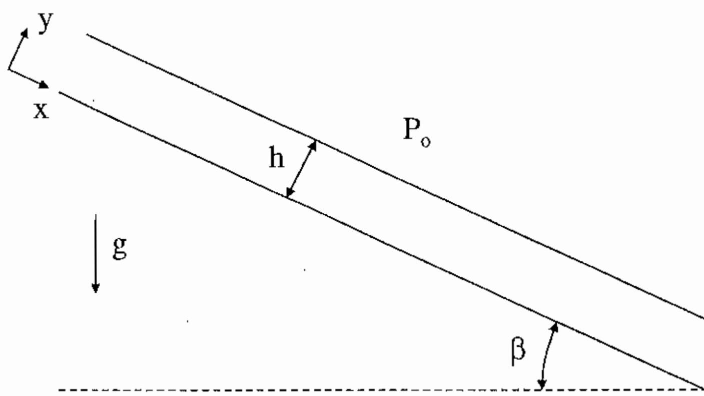
Problem 3



A stream of a frictionless liquid of height H and speed U flows over a small bump of height $\delta(x)$ where $\delta(x) \ll H$. Derive an expression for the slope of the free surface above the bump to infer whether, and under what conditions, the free surface rises or falls.

Doctoral Qualifying Examination, January 2014
Mechanical Engineering Department, Columbia University
FLUID MECHANICS

1) An incompressible, Newtonian fluid maintains a steady, two dimensional flow under the action of gravity along an inclined plane of angle β relative to the horizontal. The thickness of the fluid film is h and the pressure on the free surface is p_o.



a) Starting from the continuity and Navier-Stokes equations in three dimensions write down the governing system of equations and the appropriate boundary conditions, carefully outlining yonr assumptions.

b) Calculate the pressure and velocity fields.

c) Calcnlate the stress and strain rate tensors.

d) Calculate the film thickness in terms of the volnmetric flow rate.

2) Consider the flow of a perfect, barotropic fluid under the influence of a couservative body force. Prove Kelvin's theorem, i.e.

$$\frac{D\Gamma}{Dt} = \frac{D}{Dt}\oint_C V_i dx_i = 0$$

3) Starting with conservation of angular momentum in the absence of point torques, i.e.

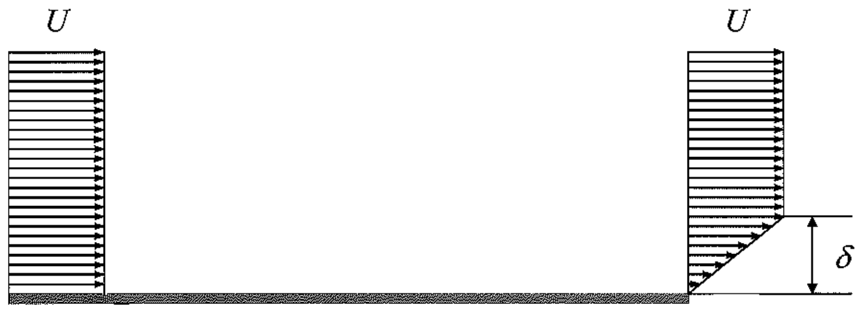
$$\int_{V(t)} \rho\, \vec{r} \times \frac{D\vec{U}}{Dt} dV = \int_{V(t)} \left[\vec{r} \times \rho\, \vec{b} - \vec{\nabla}\cdot\left(\vec{T} \times \vec{r}\right)\right] dV$$

and nsing tensor notation show that the stress tensor is symmetric.

Doctoral Qualifying Examination, January 2013
Mechanical Engineering, Columbia University
FLUID MECHANICS

Problem 1

The profiles of the horizontal component of the velocity are measured at the leading and trailing edges of a flat plate, inside a wind tunnel. The results are shown in the figure below.



Assuming steady flow and uniform pressure throughout, calculate the force (per unit depth) exerted on the plate.

5555

Doctoral Qualifying Examination, January 2013
Mechanical Engineering, Columbia University
FLUID MECHANICS

Problem 2

Derive Cauchy's equation of motion (conservation of linear momentum) using a Lagrangian (material) control volume. Next, assume a Newtonian fluid, for which,

$$\tilde{T} = 2\mu\,\tilde{D} + \lambda\left(\nabla\cdot\vec{V}\right)\tilde{I} - p\,\tilde{I}$$

where $\tilde{T}$ is the stress tensor, $\tilde{D}$ is the strain rate tensor, $\tilde{I}$ is the unit tensor, p is the pressure, $\vec{V}$ is the velocity vector, and λ and μ are the viscosity coefficients.

Derive the Navier-Stokes equation. Further simplify by assuming incompressible flow and constant fluid properties.

$$hints: \quad \frac{D}{Dt}\int_V f\, dV = \int_V \left[\frac{\partial f}{\partial t} + \vec{\nabla}\cdot\left(\vec{V}f\right)\right] dV$$

$$traction: \quad \vec{t} = \hat{n}\cdot\tilde{T}$$

Doctoral Qualifying Examination, January 2013
Mechanical Engineering, Columbia University
FLUID MECHANICS

Problem 3

Starting from the Navier-Stokes equation and assuming steady flow of an inviscid fluid write down Euler's equation and discuss the implications on the boundary conditions. Next, assuming a conservative body force, derive Bernoulli's equation.

Doctoral Qualifying Examination, January 2012
Mechanical Engineering, Columbia University
FLUID MECHANICS

1) Consider the steady, laminar flow of an incompressible, Newtonian fluid of constant viscosity μ, flowing in an infinitely long, rectangular conduit, with cross-sectional dimensions a and b. Ignoring gravity, write down the applicable system of governing equations and solve for the axial velocity distribution in terms of the axial pressure gradient and the cross-sectional coordinates, e.g., $U_z = U_z(x, y, \partial p / \partial z)$

$$\textit{hints:} \quad \frac{D\rho}{Dt} + \rho \nabla \cdot \vec{V} = 0 \quad \& \quad \rho \frac{D\vec{V}}{Dt} = -\nabla p + (\lambda + \mu)\nabla(\nabla \cdot \vec{V}) + \mu \nabla^2 \vec{V} + \rho \vec{b}$$

2) The spatial description of a continuum flow is given by,

$$x_1 = \frac{1}{2}(\xi_1 + \xi_2)e^t + \frac{1}{2}(\xi_1 - \xi_2)e^{-t}$$
$$x_2 = \frac{1}{2}(\xi_1 + \xi_2)e^t - \frac{1}{2}(\xi_1 - \xi_2)e^{-t}$$
$$x_2 = \xi_3$$

a) Show that the Jacobian does not vanish for this motion.
b) Is the flow incompressible? Does a material volume expand or shrink in time?
c) Determine the material description $\vec{\xi} = \vec{\xi}(\vec{x}, t)$.
d) Calculate the velocity and acceleration fields in terms of the material coordinates.
e) Calculate the velocity and acceleration fields in terms of the spatial coordinates.
f) Calculate the velocity gradient tensor L_{ij}.
g) Calculate the rate of deformation tensor D_{ij}.
h) Assuming a Newtonian fluid calculate the stress tensor τ_{ij}.

$$\textit{hint:} \quad \tau_{ij} = (-p + \lambda D_{kk})\delta_{ij} + 2\mu D_{ij}$$

i) Calculate the three invariants of the stress tensor.

$$\textit{hint:} \quad \Psi - \lambda \Phi + \lambda^2 \Theta - \lambda^3 = 0$$

3) Consider the two dimensional potential flow resulting from a source of strength *m* at *z=ih* and a source of strength *m* at *z=-ih*.

a) Calculate the complex potential for this flow.

$$\textit{hint:} \quad f(z) = \frac{m}{2\pi}\ln(z - z_0)$$

b) Calculate the velocity components u(x,y) and v(x,y).
c) Sketch the potential lines and the streamlines, and identify the stagnation points.
Note that the real axis is the streamline $\psi=0$. Hence, the solution is also valid for flow from a source of strength *m* at *z=ih* against an infinite plate along the real axis (*y=0*).
d) Calculate the pressure distribution along the surface of the plate.
e) Assuming that the pressure below the plate is equal to the stagnation pressure in the fluid, calculate the force exerted on the plate.

$$\textit{hint:} \quad \int \frac{x^2 dx}{(x^2 + a^2)^2} = -\frac{x}{2(x^2 + a^2)} + \frac{1}{2a}\tan^{-1}\left(\frac{x}{2a}\right)$$

2011 QUALIFYING EXAM - FLUID MECHANICS QUESTIONS

1) Show that the most general constitutive equation of a Stokesian fluid is a quadratic by proving that powers of the rate of deformation tensor higher than the second can always be expressed in terms of lower powers (Caley-Hamilton theorem).

2) The velocity and pressure fields of a continuum flow are given by,

$$\vec{V}(\vec{x},t)=\left(x_1+x_3e^{-t}\right)\hat{i}_1-2x_2\,\hat{i}_2+x_3\,\hat{i}_3$$

$$p(\vec{x},t)=p_0-\rho\left(\frac{x_1^2}{2}+2x_2^2+\frac{x_3^2}{2}+x_1x_3e^{-t}\right)$$

where p_0 is the stagnation pressure (constant).

a) Determine the spatial description $\vec{x}=\vec{x}(\vec{\xi},t)$ where $\vec{\xi}=\vec{x}(0)$.

b) Calculate and sketch the pathline passing through point (-1, 1, 1) at t = 0.

c) Calculate the acceleration field $\vec{a}(\vec{x},t)$.

d) Calculate the velocity gradient tensor L_{ij}.

e) Calculate the rate of deformation tensor D_{ij}.

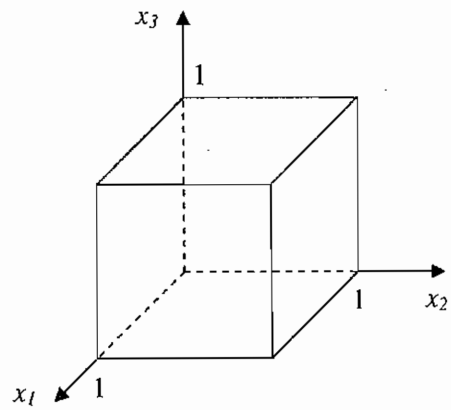
f) Is the flow incompressible?

g) Assuming a Newtonian fluid calculate the stress tensor τ_{ij}.

hint: $\tau_{ij}=\left(-p+\lambda\,D_{kk}\right)\delta_{ij}+2\mu D_{ij}$

h) Assuming constant density calculate the rate of mass inflow into the Eulerian control volume (fixed in space and time) consisting of the unit cube, as shown in the accompanying figure.

i) Calculate the total surface force acting on the side facing the positive x_1 axis.

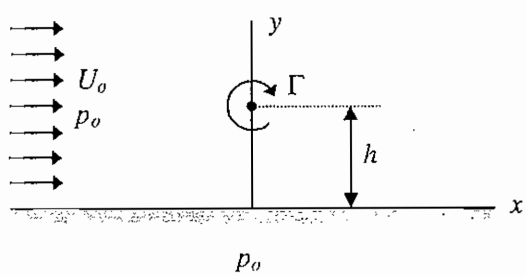


3) Consider the irrotational flow of a perfect, incompressible fluid flowing past a wall with a vortex of strength Γ a distance h away from the wall, as shown in the figure. Far away from the vortex the flow is parallel with uniform velocity U_o and free stream static pressure p_o.

a) Write down the complex potential for this flow.

b) Calculate the velocity field and verify that, far from the vortex, the flow is indeed parallel with uniform velocity U_o. Also verify that the boundary condition is satisfied.

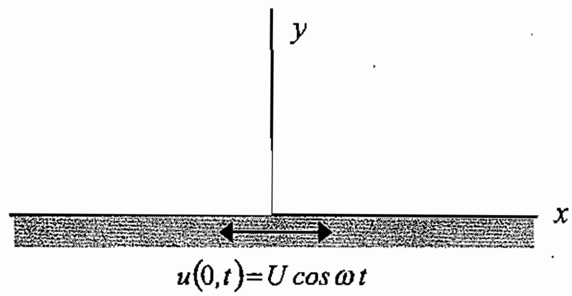
c) Calculate the pressure distribution $p(x)$ along the wall and the total force (per unit depth) on the wall assuming that the pressure on the other side of the wall is p_o.



Hint : complex potential of a vortex : $-\frac{i\Gamma}{2\pi}ln(z-z_0)$

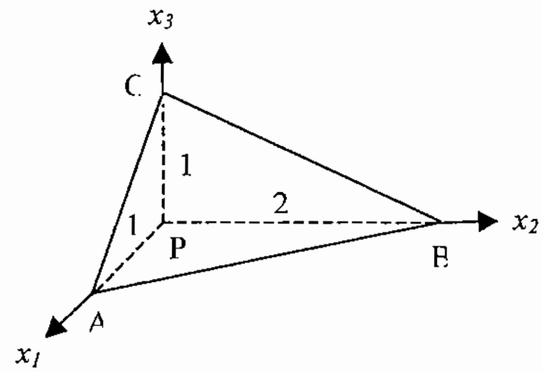
2010 QUALIFYING EXAM - FLUID MECHANICS QUESTIONS

1) Consider an infinitely long flat plate above which there is an incompressible, Newtonian fluid with viscosity μ and density ρ. The plate is oscillating along the x axis with a frequency ω as shown in the figure. Calculate the resulting velocity field of the fluid.



2) The stress tensor at point P is,

$$\tau_{rs} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$



a) Find the unit vector (outward) normal to the plane ABC of the tetrahedron shown in the figure.

b) Calculate the traction vector on the plane passing through point P and parallel to plane ABC.

c) Calculate the component of the traction vector normal to the plane.

d) Calculate the angle between the traction vector and the normal to the plane.

e) Calculate the three invariants of the stress tensor.

hint: $\Psi - \lambda\Phi + \lambda^2\Theta - \lambda^3 = 0$

3) The particle paths of Gerstner waves are given by,

$$x_1 = a + \frac{e^{-bk}}{k} sin[k(a+ct)]$$

$$x_2 = -b - \frac{e^{-bk}}{k} cos[k(a+ct)]$$

$$x_3 = constant$$

a) Relate the constants a and b to the initial position and show that the particle paths are circles.

b) Find the velocity vector.

c) Show that the Jacobian is equal to 1.

421

FLUID MECHANICS

Problem 1:

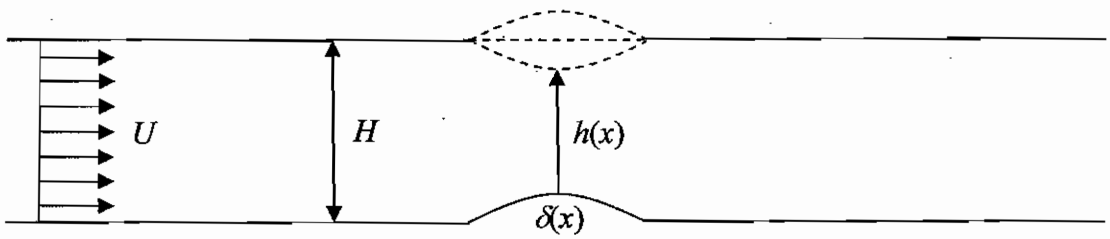
Starting from the Navier-Stokes equation and assuming steady flow of an inviscid fluid subject to a conservative body force, derive Bernoulli's equation.

422

FLUID MECHANICS

Problem 2:

A stream of a frictionless liquid of height H and speed U flows over a small bump of height $\delta(x)$ where $\delta(x) << H$, as shown in the figure below.



Derive an expression for the slope of the free surface above the bump to infer whether, and under what conditions, the free surface rises or falls.

423

FLUID MECHANICS

Problem 3:

The velocity field of a continuum flow is given by,

$$\vec{V}(\vec{x},t)=\left(x_1+x_3e^{-t}\right)\hat{i}_1-2x_2\,\hat{i}_2+x_3\,\hat{i}_3$$

a) Determine the spatial description $\vec{x}=\vec{x}\left(\vec{\xi},t\right)$ where $\vec{\xi}=\vec{x}(0)$.

b) Calculate and sketch the pathline passing through point (-1, 1, 1) at time t = 0.

c) Calculate the acceleration field $\vec{a}(\vec{x},t)$.

d) Calculate the rate of deformation tensor D_{ij}.

e) Is the flow incompressible?

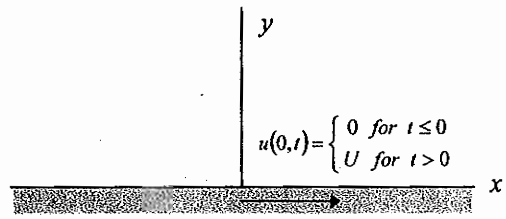
f) Assuming a Newtonian fluid calculate the stress tensor τ_{ij}.

Hint: $\tau_{ij}=\left(-p+\lambda\,D_{kk}\right)\delta_{ij}+2\mu D_{ij}$

g) Calculate the principal values of τ_{ij}.

2008 QUALIFYING EXAM - FLUID MECHANICS QUESTIONS

1) Consider an infinitely long flat plate above which there is a Newtonian fluid. Initially, both the plate and the fluid are at rest. Then, suddenly, the plate begins to move with constant velocity as shown in the figure. Calculate the resulting velocity field of the fluid.



2) The constitutive relation of a Newtonian fluid is known to be $\tilde{P} = \alpha\tilde{D} + \beta\tilde{I}$ where $\tilde{P}$ is the viscous stress tensor, $\tilde{D}$ is the strain rate tensor, $\tilde{I}$ is the unit tensor, and α and β are constants. Measurements, however, reveal that the constitutive relation of this fluid is actually $\tilde{P} = \gamma\tilde{D}^4 + \tilde{D}^3$, where γ is a measured constant. Is this possible? If so, how are the constants α, β and γ related to the invariants of the strain rate tensor?

$$hint: \quad \Psi - \lambda\Phi + \lambda^2\Theta - \lambda^3 = 0$$

3) Derive Cauchy's equation of motion (conservation of linear momentum) using an Eulerian control volume (fixed in space). Next, assume a Newtonian fluid, for which,

$$\tilde{T} = 2\mu\,\tilde{D} + \lambda\left(\nabla\cdot\vec{V}\right)\tilde{I} - p\,\tilde{I}$$

where $\tilde{T}$ is the stress tensor, $\tilde{D}$ is the strain rate tensor, $\tilde{I}$ is the unit tensor, p is the pressure, $\vec{V}$ is the velocity vector, and λ and μ are the viscosity coefficients.

Derive the Navier-Stokes equation. Further simplify by assuming incompressible flow and constant fluid properties.

$$hints: \quad flux\ of\ f = \int_S f\,\vec{V}\cdot\hat{n}\,dS$$

$$traction\ \vec{t} = \hat{n}\cdot\tilde{T}$$

FLUID MECHANICS

Doctoral Qualifying Examination, January 2007
Mechanical Engineering, Columbia University

1) Consider the flow of a perfect, barotropic fluid under the influence of a conservative body force. Prove Kelvin's theorem, i.e.,

$$\frac{D\Gamma}{Dt} = \frac{D}{Dt}\oint_C V_i dx_i = 0$$

2) It is observed that the mass flow rate from a circular pipe is $\dot{m}$ and that the flow is laminar. It is necessary to design a new pipe to accommodate a new mass flow rate $\dot{m}/16$ of the same fluid, using the same (constant) pressure gradient. What is the required change in pipe radius?

Hint : Navier – Stokes equations in cylindrical coordinates

$$\rho\left(\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta}-\frac{u_\theta^2}{r}+u_z\frac{\partial u_r}{\partial z}\right)=-\frac{\partial p}{\partial r}+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_r}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2}+\frac{\partial^2 u_r}{\partial z^2}-\frac{u_r}{r^2}-\frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right]$$

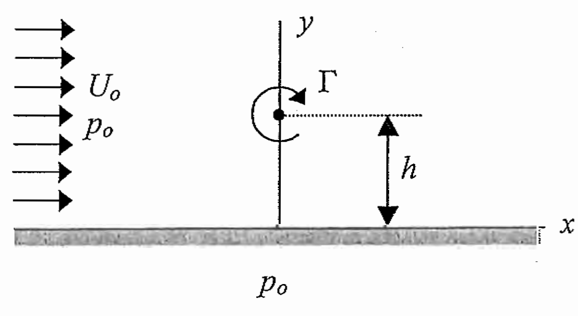
$$\rho\left(\frac{\partial u_\theta}{\partial t}+u_r\frac{\partial u_\theta}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{u_r u_\theta^2}{r}+u_z\frac{\partial u_\theta}{\partial z}\right)=-\frac{1}{r}\frac{\partial p}{\partial \theta}+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_\theta}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2}+\frac{\partial^2 u_\theta}{\partial z^2}-\frac{u_\theta}{r^2}+\frac{2}{r^2}\frac{\partial u_r}{\partial \theta}\right]$$

$$\rho\left(\frac{\partial u_z}{\partial t}+u_r\frac{\partial u_z}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta}+u_z\frac{\partial u_z}{\partial z}\right)=-\frac{\partial p}{\partial z}+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2}+\frac{\partial^2 u_z}{\partial z^2}\right]$$

3) Consider the flow of a perfect, incompressible and irrotational fluid flowing past a wall with a vortex of strength Γ a distance h away from the wall, as shown in the figure. Far away from the vortex the flow is parallel with uniform velocity U_o and free stream static pressure p_o.

a) Write down the complex potential for this flow.

b) Calculate the velocity field and verify that, far from the vortex, the flow is indeed parallel with uniform velocity U_o. Also verify that the boundary condition is satisfied.



c) Calculate the pressure distribution $p(x)$ along the wall and the total force (per unit depth) on the wall assuming that the pressure on the other side of the wall is p_o

Hint : complex potential of a free vortex : $-\dfrac{i\Gamma}{2\pi}\ln(z-z_0)$

FLUID MECHANICS

Doctoral Qualifying Examination, January 2006
Mechanical Engineering, Columbia University

Problem 1:

The velocity field of a continuum flow of a constant density fluid is given by,

$$\vec{V}(\vec{x},t) = (1+t)\,x_1\,\hat{i}_1 - x_2\,\hat{i}_2$$

a. Calculate the equation of the streamline passing through point (1, 1, 1).

b. Calculate the equation of the pathline passing through point (1, 1, 1) at time t = 0.

c. Calculate the acceleration field $\vec{a}(\vec{x},t)$.

d. Is the flow incompressible?

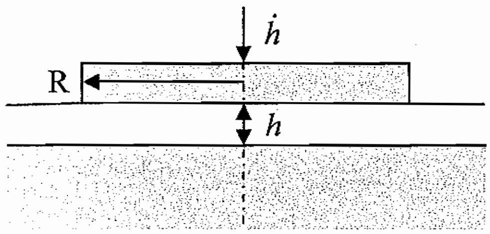
FLUID MECHANICS

Doctoral Qualifying Examination, January 2006
Mechanical Engineering, Columbia University

Problem 2:

A circular disk of radius R slowly squeezes a thin liquid film at a rate of $\dot{h}$ as shown in the figure. Assuming incompressible flow calculate:

a. The velocity field.
b. The pressure distribution.
c. The force required to move the disk.



Hint : Navier – Stokes equations in cylindrical coordinates

$$\rho\left(\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta}-\frac{u_\theta^2}{r}+u_z\frac{\partial u_r}{\partial z}\right)=-\frac{\partial p}{\partial r}+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_r}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2}+\frac{\partial^2 u_r}{\partial z^2}-\frac{u_r}{r^2}-\frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right]$$

$$\rho\left(\frac{\partial u_\theta}{\partial t}+u_r\frac{\partial u_\theta}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{u_r u_\theta^2}{r}+u_z\frac{\partial u_\theta}{\partial z}\right)=-\frac{1}{r}\frac{\partial p}{\partial \theta}+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_\theta}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2}+\frac{\partial^2 u_\theta}{\partial z^2}-\frac{u_\theta}{r^2}+\frac{2}{r^2}\frac{\partial u_r}{\partial \theta}\right]$$

$$\rho\left(\frac{\partial u_z}{\partial t}+u_r\frac{\partial u_z}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta}+u_z\frac{\partial u_z}{\partial z}\right)=-\frac{\partial p}{\partial z}+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2}+\frac{\partial^2 u_z}{\partial z^2}\right]$$

FLUID MECHANICS

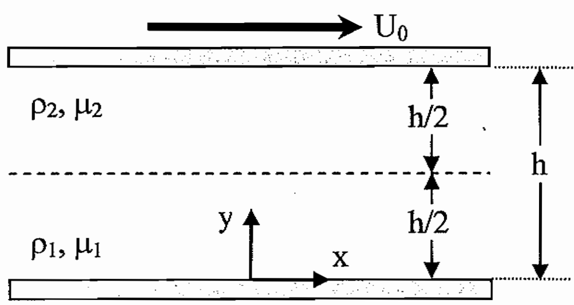
Doctoral Qualifying Examination, January 2006
Mechanical Engineering, Columbia University

Problem 3:

Consider the steady laminar incompressible flow between two infinite parallel plates as shown in the figure. The upper plate moves with constant velocity U_0 to the right and the lower plate is stationary. The lower half of the region between the plates ($0 \leq y \leq h/2$) is filled with fluid with density ρ_1 and viscosity μ_1 and the upper half ($h/2 \leq y \leq h$) is filled with fluid with density ρ_2 and viscosity μ_2.

a. Calculate and sketch the velocity profile.

b. Calculate and sketch the shear stress profile.



FLUID MECHANICS

Doctoral Qualifying Examination, January 2005
Mechanical Engineering, Columbia University

Problem 1

An oil having a surface tension of 0.080 N/m is sprayed at a rate of 50 liters per minute. Ignoring thermal and frictional effects, find the power necessary for a pump to produce droplets of 1 μm in diameter at this rate.

Problem 2

A plate 1 m wide and 1 m long is placed in a uniform stream of air of velocity 20 m/s. The velocity profile at the end of the plate is given by
$u/U=2\eta-\eta^2$ with $\eta=y/\delta$ and δ=3mm.
Using the appropriate control volume, find the force acting on the plate in the air stream direction.

Problem 3

The horizontal component of the velocity in a boundary layer (laminar, incompressible, 2 dimensional) is given by:
$u=U(2y/\delta-y^2/\delta^2)$ $y\leq\delta$ with $\delta=Cx^{1/2}$ C=constant
Find v(x,y) for $y\leq\delta$. What is the maximum value of v for x=1m when U=3m/s and δ=.11 m.

280

FLUID MECHANCICS

Doctoral Qualifying Examination , January 2004
Mechanical Engineering, Columbia University

1. Give the conditions under which the equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u_i)}{\partial x_i} \quad \text{reduces to} \quad \frac{\partial u_i}{\partial x_i} = 0$$

Under which conditions could the mass flux through a closed surface within the fluid be different from zero if the conservation of mass is given by:

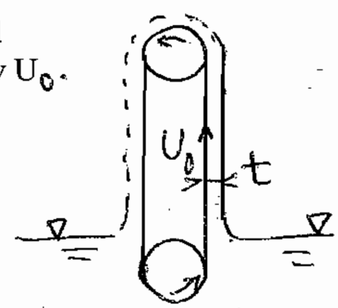
$$\frac{dm}{dt} = \frac{\partial}{\partial t}\int_{\forall} \rho \, d\forall + \int_{CS} \rho \, \vec{V} \cdot d\vec{A}$$

2. The stream function of a certain two dimensional flow field is

$$\psi = kxy$$

Show that the flow field is irrotational and find the corresponding velocity potential. What is the maximum pressure if the pressure at the point of coordinates (1,1) is zero?

3. A liquid (viscosity μ, density ρ) film of thickness t, is established on a vertical, two dimensional belt moving with upward velocity U_0. Find the velocity distribution and the volume flow rate.



4. Axial flow occurs between two circular cylinders for the geometry sketched. Draw as precisely as you can the velocity profile within the annulus as the inner cylinder moves steadily down with U_0.
Do not solve the Navier-Stokes equations but give as much information as possible about this flow.