ELASTICITY
Doctoral Qualifying Examination, May 2016
Mechanical Engineering Department, Columbia University

Problem 1

The components of a stress tensor $\mathbf{T}$ at some point P are given by

$$[\mathbf{T}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -4 \end{bmatrix}$$

a) Find the invariants of $\mathbf{T}$.
b) Find the maximum shearing stress and the planes on which it acts.
c) Find the normal stress on these planes.
d) Prove that there are no planes passing through P on which the normal stress is +4.

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Problem 2

The state of stress in a prismatic bar aligned along the x_3-axis is given by

$$[\mathbf{T}] = \begin{bmatrix} 0 & 0 & \dfrac{\partial \phi}{\partial x_2} \\ 0 & 0 & -\dfrac{\partial \phi}{\partial x_1} \\ \dfrac{\partial \phi}{\partial x_2} & -\dfrac{\partial \phi}{\partial x_1} & 0 \end{bmatrix}$$

where

$$\phi = \frac{32\mu\theta a^2}{\pi^3} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n^3}(-1)^{(n-1)/2}\left[1 - \frac{\cosh(n\pi x_2/2a)}{\cosh(n\pi b/2a)}\right]\cos\frac{n\pi x_1}{2a}$$

In this expression, μ represents the shear modulus and θ, a and b are constants.

(a) Show that this state of stress satisfies stress equilibrium in the absence of body forces.

(b) Evaluate the stress components T_{13} and T_{23} from the above expressions.

(c) Evaluate the traction vector $\mathbf{t}$ on planes parallel to the x_3-axis, with outward normal $\mathbf{n} = n_1\mathbf{e}_1 + n_2\mathbf{e}_2$.

(d) By inspection, determine the planes on which the traction $\mathbf{t}$ from your previous answer reduces to $\mathbf{0}$. Based on these results, what is the shape of the cross-section of this prismatic bar, given that its lateral boundary is traction-free?

(e) Evaluate the net force and net moment on the cross-section $x_3 =$ constant. Based on these results, what is the nature of the loading of this prismatic bar?

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Problem 3

Differential operators in cylindrical coordinates may be found in the Hints section on the last page.

Set up St-Venant's torsion problem for a prismatic bar using cylindrical coordinates in the basis $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$. St-Venant assumed that all cross-sections along the longitudinal $z-$axis rotate as rigid bodies but do not necessarily remain plane. If they warp, all sections will warp the same way. The angle of rotation α of the cross-section at z is given by $\alpha = \Theta z$, where Θ is the angle of twist per unit length and $\alpha \ll 1$ for infinitesimal deformations.

a) Express the general form of the displacement field $\mathbf{u}(r,\theta,z)$ in cylindrical coordinates (call the warping function φ). Clearly indicate the functional dependence of φ.
b) Using this expression for $\mathbf{u}$, evaluate the infinitesimal strain tensor $\mathbf{E}$.
c) Assuming that the bar is linear isotropic elastic, evaluate the Cauchy stress tensor $\mathbf{T}$.
d) Satisfy the balance of linear momentum. What is the governing equation for the warping function?
e) What is the boundary condition on the warping function?
f) Show that letting $\varphi = 0$ produces the torsional response for a bar with circular cross-section.
g) Evaluate $\mathbf{T}$ and $\mathbf{E}$ for this special case.

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Problem 4

The axiom of entropy inequality states that

$$-\rho\left(\frac{D\psi}{Dt}+\eta\frac{D\theta}{Dt}\right)+\mathbf{T}:\mathbf{D}-\frac{1}{\theta}\mathbf{q}\cdot\operatorname{grad}\theta\geq 0$$

where ρ is the mass density, θ is the temperature, $\mathbf{D}$ is the rate of deformation tensor (the symmetric part of the velocity gradient), ψ is the specific free energy, η is the specific entropy, $\mathbf{T}$ is the Cauchy stress, and $\mathbf{q}$ is the heat flux.

a) When formulating constitutive relations for the functions of state of a solid, explain why it makes no sense to include ρ in the list of state variables, once the deformation gradient $\mathbf{F}$ has been included.

b) Let the list of state variables for a particular analysis be restricted to $(\theta,\mathbf{F},\mathbf{D})$. Find the constraints imposed on the functions of state $(\psi,\eta,\mathbf{T},\mathbf{q})$ by the entropy inequality.

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Hints

Cylindrical coordinates

Gradient of a scalar field ϕ

$$\operatorname{grad}\phi = \frac{\partial \phi}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial \phi}{\partial \theta}\mathbf{e}_\theta + \frac{\partial \phi}{\partial z}\mathbf{e}_z$$

Gradient of a vector field $\mathbf{v}$

$$[\operatorname{grad}\mathbf{v}] = \begin{bmatrix} \dfrac{\partial v_r}{\partial r} & \dfrac{1}{r}\left(\dfrac{\partial v_r}{\partial \theta} - v_\theta\right) & \dfrac{\partial v_r}{\partial z} \\ \dfrac{\partial v_\theta}{\partial r} & \dfrac{1}{r}\left(\dfrac{\partial v_\theta}{\partial \theta} + v_r\right) & \dfrac{\partial v_\theta}{\partial z} \\ \dfrac{\partial v_z}{\partial r} & \dfrac{1}{r}\dfrac{\partial v_z}{\partial \theta} & \dfrac{\partial v_z}{\partial z} \end{bmatrix}$$

Divergence of a vector field $\mathbf{v}$

$$\operatorname{div}\mathbf{v} = \frac{\partial v_r}{\partial r} + \frac{1}{r}\left(\frac{\partial v_\theta}{\partial \theta} + v_r\right) + \frac{\partial v_z}{\partial z}$$

Laplacian of a vector field $\mathbf{v}$

$$\begin{aligned} \operatorname{lap}\mathbf{v} = &\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial (r v_r)}{\partial r}\right) + \frac{1}{r^2}\left(\frac{\partial^2 v_r}{\partial \theta^2} - 2\frac{\partial v_\theta}{\partial \theta}\right) + \frac{\partial^2 v_r}{\partial z^2}\right)\mathbf{e}_r \\ &+ \left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial (r v_\theta)}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{\partial^2 v_\theta}{\partial z^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta}\right)\mathbf{e}_\theta \\ &+ \left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right)\mathbf{e}_z \end{aligned}$$

Divergence of a tensor field $\mathbf{T}$

$$\begin{aligned} \operatorname{div}\mathbf{T} = &\left(\frac{\partial T_{rr}}{\partial r} + \frac{1}{r}\left(\frac{\partial T_{r\theta}}{\partial \theta} + T_{rr} - T_{\theta\theta}\right) + \frac{\partial T_{rz}}{\partial z}\right)\mathbf{e}_r \\ &+ \left(\frac{\partial T_{\theta r}}{\partial r} + \frac{1}{r}\left(\frac{\partial T_{\theta\theta}}{\partial \theta} + T_{\theta r} + T_{r\theta}\right) + \frac{\partial T_{\theta z}}{\partial z}\right)\mathbf{e}_\theta \\ &+ \left(\frac{\partial T_{zr}}{\partial r} + \frac{1}{r}\left(\frac{\partial T_{z\theta}}{\partial \theta} + T_{zr}\right) + \frac{\partial T_{zz}}{\partial z}\right)\mathbf{e}_z \end{aligned}$$

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Problem 1

a) State the Cayley-Hamilton theorem for second-order tensors in 3D.

b) Define the invariants I_1, I_2 and I_3 of the tensor $\mathbf{A}$.

c) Derive the tensorial expressions for $\partial I_1/\partial \mathbf{A}$, $\partial I_2/\partial \mathbf{A}$ and $\partial I_3/\partial \mathbf{A}$. (Do not assume that $\mathbf{A}$ is symmetric.)

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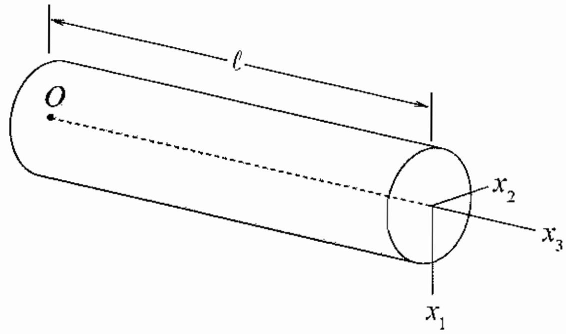
Problem 2



A prismatic bar of length ℓ, with circular cross-section of radius a, is subjected to an unspecified loading configuration. The Cartesian components of the displacement field $\mathbf{u}(x_1,x_2,x_3)$ for this problem are given by

$$u_1 = \frac{\nu P(\ell - x_3)}{2EI}\left(x_1^2 - x_2^2\right) + \frac{Px_3^2}{6EI}(3\ell - x_3)$$

$$u_2 = \frac{\nu P(\ell - x_3)}{EI}x_1x_2$$

$$u_3 = -\frac{Px_1x_3(2\ell - x_3)}{2EI} + \frac{Px_1}{4EI}\left[(3+2\nu)a^2 - x_1^2 - x_2^2\right]$$

where the origin is at O. E is Young's modulus, ν is Poisson's ratio, $I = \pi a^4/4$ is the area moment of inertia of the cross-section about the x_2 axis, and P is a constant.

(a) Evaluate the components of the infinitesimal strain tensor.
(b) Evaluate the components of the stress tensor.
(c) Verify that this state of stress satisfies the balance of linear momentum under static conditions, in the absence of body forces.
(d) What is the traction on the lateral boundary?
(e) What is the traction on the circular cross-section located at $x_3 = \ell$? What is the net force acting on this cross-section? What is the net moment about the centroid of this cross-section?
(f) What is the traction on the circular cross-section located at $x_3 = 0$? What is the net force acting on this cross-section? What is the net moment about the centroid of this cross-section?
(g) Evaluate the displacement $\mathbf{u}(0,0,x_3)$ of the centerline and sketch the deformed centerline in the range $0 \le x_3 \le \ell$.
(h) Based on these results, what type of loading is this bar subjected to? How would you apply St-Venant's principle to this problem?

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Problem 3

Differential operators in cylindrical coordinates may be found in the Hints section on the last page.

Set up St-Venant's torsion problem for a prismatic bar using cylindrical coordinates in the basis $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$. St-Venant assumed that all cross-sections along the longitudinal $z-$axis rotate as rigid bodies but do not necessarily remain plane. If they warp, all sections will warp the same way. The angle of rotation α of the cross-section at z is given by $\alpha = \Theta z$, where Θ is the angle of twist per unit length and $\alpha \ll 1$ for infinitesimal deformations.

a) Express the general form of the displacement field $\mathbf{u}(r,\theta,z)$ in cylindrical coordinates (call the warping function φ). Clearly indicate the functional dependence of φ.
b) Using this expression for $\mathbf{u}$, evaluate the infinitesimal strain tensor $\mathbf{E}$.
c) Assuming that the bar is linear isotropic elastic, evaluate the Cauchy stress tensor $\mathbf{T}$.
d) Satisfy the balance of linear momentum. What is the governing equation for the warping function?
e) What is the boundary condition on the warping function?
f) Show that letting $\varphi = 0$ produces the torsional response for a bar with circular cross-section.
g) Evaluate $\mathbf{T}$ and $\mathbf{E}$ for this special case.

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Problem 4

Prove that stress is a tensor.

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Hints

Cylindrical coordinates

Gradient of a scalar field ϕ

$$\operatorname{grad}\phi = \frac{\partial \phi}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial \phi}{\partial \theta}\mathbf{e}_\theta + \frac{\partial \phi}{\partial z}\mathbf{e}_z$$

Gradient of a vector field $\mathbf{v}$

$$[\operatorname{grad}\mathbf{v}] = \begin{bmatrix} \dfrac{\partial v_r}{\partial r} & \dfrac{1}{r}\left(\dfrac{\partial v_r}{\partial \theta} - v_\theta\right) & \dfrac{\partial v_r}{\partial z} \\ \dfrac{\partial v_\theta}{\partial r} & \dfrac{1}{r}\left(\dfrac{\partial v_\theta}{\partial \theta} + v_r\right) & \dfrac{\partial v_\theta}{\partial z} \\ \dfrac{\partial v_z}{\partial r} & \dfrac{1}{r}\dfrac{\partial v_z}{\partial \theta} & \dfrac{\partial v_z}{\partial z} \end{bmatrix}$$

Divergence of a vector field $\mathbf{v}$

$$\operatorname{div}\mathbf{v} = \frac{\partial v_r}{\partial r} + \frac{1}{r}\left(\frac{\partial v_\theta}{\partial \theta} + v_r\right) + \frac{\partial v_z}{\partial z}$$

Laplacian of a vector field $\mathbf{v}$

$$\begin{aligned} \operatorname{lap}\mathbf{v} = &\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial (r v_r)}{\partial r}\right) + \frac{1}{r^2}\left(\frac{\partial^2 v_r}{\partial \theta^2} - 2\frac{\partial v_\theta}{\partial \theta}\right) + \frac{\partial^2 v_r}{\partial z^2}\right)\mathbf{e}_r \\ &+ \left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial (r v_\theta)}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{\partial^2 v_\theta}{\partial z^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta}\right)\mathbf{e}_\theta \\ &+ \left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right)\mathbf{e}_z \end{aligned}$$

Divergence of a tensor field $\mathbf{T}$

$$\begin{aligned} \operatorname{div}\mathbf{T} = &\left(\frac{\partial T_{rr}}{\partial r} + \frac{1}{r}\left(\frac{\partial T_{r\theta}}{\partial \theta} + T_{rr} - T_{\theta\theta}\right) + \frac{\partial T_{rz}}{\partial z}\right)\mathbf{e}_r \\ &+ \left(\frac{\partial T_{\theta r}}{\partial r} + \frac{1}{r}\left(\frac{\partial T_{\theta\theta}}{\partial \theta} + T_{\theta r} + T_{r\theta}\right) + \frac{\partial T_{\theta z}}{\partial z}\right)\mathbf{e}_\theta \\ &+ \left(\frac{\partial T_{zr}}{\partial r} + \frac{1}{r}\left(\frac{\partial T_{z\theta}}{\partial \theta} + T_{zr}\right) + \frac{\partial T_{zz}}{\partial z}\right)\mathbf{e}_z \end{aligned}$$

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Problem 1

The deformation gradient $\mathbf{F}$ is related to the displacement vector $\mathbf{u}$ via $\mathbf{F} = \mathbf{I} + \mathrm{Grad}\,\mathbf{u}$, where $\mathrm{Grad}\,\mathbf{u} = \partial\mathbf{u}/\partial\mathbf{X}$ is the material gradient of $\mathbf{u}$. The Lagrangian strain tensor $\mathbf{E}$ is related to the deformation gradient via

$$\mathbf{E} = \frac{1}{2}\left(\mathbf{F}^T \cdot \mathbf{F} - \mathbf{I}\right)$$

where $\mathbf{I}$ is the identity tensor.

Consider the limit of infinitesimal strains and rotations, where we may set $\mathrm{Grad}\,\mathbf{u} \approx \mathrm{grad}\,\mathbf{u}$, where $\mathrm{grad}\,\mathbf{u}$ is the spatial gradient of $\mathbf{u}$.

(a) Show that the deformation gradient may be written as $\mathbf{F} \approx \mathbf{I} + \boldsymbol{\varepsilon} + \boldsymbol{\omega}$, where $\boldsymbol{\varepsilon}$ is a symmetric tensor and $\boldsymbol{\omega}$ is an anti-symmetric tensor. Relate these tensors to $\mathrm{grad}\,\mathbf{u}$. (4 pts)

(b) Prove that $\mathbf{E}$ reduces to $\boldsymbol{\varepsilon}$ in this limit. (3 pts)

(c) Prove that $\mathbf{I} + \boldsymbol{\omega}$ is a rotation tensor in this limit. (3 pts)

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Problem 2

(a) Derive the axiom of linear momentum balance from first principles, starting from the integral form and ending in the differential form. Define each function of state appearing in this relation. (5 pts)

(b) Derive the axiom of angular momentum balance from first principles, starting from the integral form and ending in the differential form. Define each function of state appearing in this relation. (5 pts)

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Problem 3

The state of stress in a prismatic bar aligned along the x_3 – axis is given by

$$[\mathbf{T}] = \begin{bmatrix} 0 & 0 & \dfrac{\partial\phi}{\partial x_2} \\ 0 & 0 & -\dfrac{\partial\phi}{\partial x_1} \\ \dfrac{\partial\phi}{\partial x_2} & -\dfrac{\partial\phi}{\partial x_1} & 0 \end{bmatrix}$$

where

$$\phi = \frac{32\mu\theta a^2}{\pi^3} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n^3}(-1)^{(n-1)/2}\left[1 - \frac{\cosh(n\pi x_2/2a)}{\cosh(n\pi b/2a)}\right]\cos\frac{n\pi x_1}{2a}$$

In this expression, μ represents the shear modulus and θ, a and b are constants.

(a) Show that this state of stress satisfies stress equilibrium in the absence of body forces. (1 pt)

(b) Evaluate the stress components T_{13} and T_{23} from the above expressions. (2 pts)

(c) Evaluate the traction vector $\mathbf{t}$ on planes parallel to the x_3 – axis, with outward normal $\mathbf{n} = n_1\mathbf{e}_1 + n_2\mathbf{e}_2$. (1 pt)

(d) By inspection, determine the planes on which the traction $\mathbf{t}$ from your previous answer reduces to $\mathbf{0}$. Based on these results, what is the shape of the cross-section of this prismatic bar, given that its lateral boundary is traction-free? (2 pts)

(e) Evaluate the net force and net moment on the cross-section $x_3 =$ constant. Based on these results, what is the nature of the loading of this prismatic bar? (4 pts)

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Problem 1

Consider the following motion:

$$x_1 = X_1 + \gamma X_2$$
$$x_2 = X_2 \qquad ,$$
$$x_3 = X_3$$

where γ is a constant.

1. Evaluate the deformation gradient $\mathbf{F}$.
2. Evaluate the right Cauchy-Green tensor $\mathbf{C}$.
3. Find the invariants of $\mathbf{C}$. Does this deformation cause a volume change?
4. Consider a prismatic bar with its long axis along $\mathbf{e}_3$, and with a square 1×1 cross-section in the $X_1 - X_2$ plane. Sketch the cross-section in the reference configuration (use dashed lines) and the deformed configuration (use solid lines), when $\gamma = 1/2$. (You can put one of the corners of the square at the origin.)
5. Evaluate the infinitesimal strain tensor $\mathbf{E}$ when $\gamma \ll 1$.

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Problem 2

1. Derive the axiom of energy balance from first principles, starting from the integral form and ending in the differential form. Define each function of state appearing in this relation.
2. Derive the axiom of entropy inequality from first principles, starting from the integral form and ending in the differential form. Define each function of state appearing in this relation.
3. Using $\psi = \varepsilon - \theta\eta$, where $\psi =$ free energy per mass, $\varepsilon =$ internal energy per mass, $\eta =$ entropy per mass and $\theta =$ absolute temperature, reformulate the axiom of entropy inequality in terms of the time derivative of ψ.

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Problem 3

Present the analysis of pure bending of a beam for a linear isotropic elastic solid undergoing infinitesimal strains.

1. Define the type of geometry and boundary conditions for which this analysis applies. Sketch an example and label the sketch.
2. What is the starting assumption (initial guess) about the solution?
3. Use the balance of linear momentum, Hooke's law, and the equations of strain compatibility (see below) to solve for the general solution.
4. Use the boundary conditions to constrain the integration constants obtained in the previous step.

Your final solution should include the complete state of stress and the complete state of strain, both in matrix form.

Hints:

Hooke's law

$$\mathbf{E} = \frac{1}{E_Y}\left[(1+\nu)\mathbf{T} - \nu(\mathrm{tr}\,\mathbf{T})\mathbf{I}\right]$$

Equations of strain compatibility

$$\frac{\partial^2 E_{11}}{\partial x_2^2} + \frac{\partial^2 E_{22}}{\partial x_1^2} = 2\frac{\partial^2 E_{12}}{\partial x_1 \partial x_2}, \qquad \frac{\partial^2 E_{11}}{\partial x_2 \partial x_3} = \frac{\partial}{\partial x_1}\left(-\frac{\partial E_{23}}{\partial x_1} + \frac{\partial E_{31}}{\partial x_2} + \frac{\partial E_{12}}{\partial x_3}\right)$$

$$\frac{\partial^2 E_{22}}{\partial x_3^2} + \frac{\partial^2 E_{33}}{\partial x_2^2} = 2\frac{\partial^2 E_{23}}{\partial x_2 \partial x_3}, \qquad \frac{\partial^2 E_{22}}{\partial x_3 \partial x_1} = \frac{\partial}{\partial x_2}\left(-\frac{\partial E_{31}}{\partial x_2} + \frac{\partial E_{12}}{\partial x_3} + \frac{\partial E_{23}}{\partial x_1}\right)$$

$$\frac{\partial^2 E_{33}}{\partial x_1^2} + \frac{\partial^2 E_{11}}{\partial x_3^2} = 2\frac{\partial^2 E_{13}}{\partial x_1 \partial x_3}, \qquad \frac{\partial^2 E_{33}}{\partial x_1 \partial x_2} = \frac{\partial}{\partial x_3}\left(-\frac{\partial E_{12}}{\partial x_3} + \frac{\partial E_{23}}{\partial x_1} + \frac{\partial E_{31}}{\partial x_2}\right)$$

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Problem 1

A tensor $\mathbf{T}$ may be expressed in terms of its Cartesian components T_{ij} in the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ as $\mathbf{T} = T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$, where $\otimes$ represents the dyadic product of vectors.

a) If $\mathbf{T}$ is symmetric, express $\mathbf{T}$ in terms of its components in the Cartesian basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ where $\mathbf{v}_i$'s are the eigenvectors of $\mathbf{T}$.

b) Prove that the eigenvectors of $\mathbf{T}$ are also eigenvectors of $\mathbf{T}^{-1}$.

c) Using the results of (a) and (b), express $\mathbf{T}^{-1}$ in terms of its components in the basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

d) Verify from the results of (a) and (c) that $\mathbf{T} \cdot \mathbf{T}^{-1} = \mathbf{I}$.

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Problem 2

a) What is the definition of a second-order tensor?
b) Using the free-body-diagram analysis of a small tetrahedral region inside a loaded body, prove that the stress is a second-order tensor.

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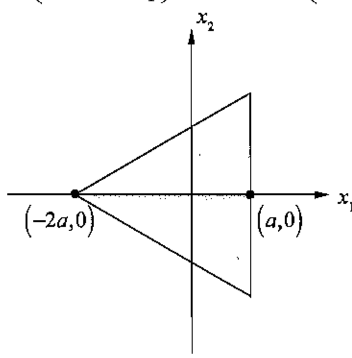
Problem 3

Consider the torsion of a prismatic bar with an equilateral triangular cross-section as shown below and recall from the St-Venant torsion analysis that the only non-zero stress components are

$$T_{13} = \mu\theta\left(-x_2 + \frac{\partial\varphi}{\partial x_1}\right), \quad T_{23} = \mu\theta\left(x_1 + \frac{\partial\varphi}{\partial x_2}\right)$$



a) Show that the warping function $\varphi = \alpha\left(3x_1^2 x_2 - x_2^3\right)$ generates an equilibrium stress field.

b) Determine the constant α in order to satisfy the traction-free lateral boundary condition. Demonstrate that the entire lateral surface is traction-free. *Hint:* You must figure out the equations of the three lines forming the boundaries of the triangle.

c) Write out explicitly the stress distribution generated by this warping function. Evaluate the magnitude τ of the shear traction at the triangular corners and along the line $x_2 = 0$ in a cross-section. Along the line $x_2 = 0$, where does the greatest value of τ occur?

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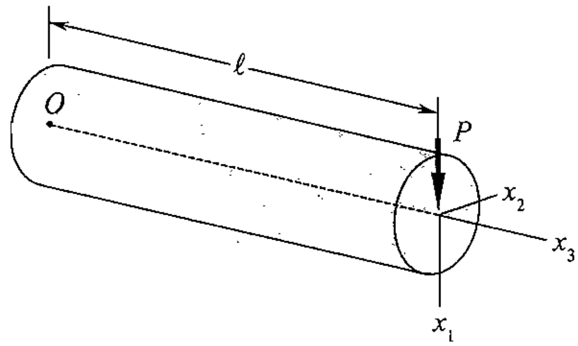
Problem 1



A prismatic beam of length ℓ, with circular cross-section of radius a, is subjected to a terminal load P as shown. The Cartesian components of the displacement field $\mathbf{u}(x_1,x_2,x_3)$ for this problem are given by

$$
\begin{aligned}
u_1 &= \frac{\nu P(\ell - x_3)}{2EI}\left(x_1^2 - x_2^2\right) + \frac{Px_3^2}{6EI}(3\ell - x_3) \\
u_2 &= \frac{\nu P(\ell - x_3)}{EI} x_1 x_2 \\
u_3 &= -\frac{Px_1x_3(2\ell - x_3)}{2EI} + \frac{Px_1}{4EI}\left[(3+2\nu)a^2 - x_1^2 - x_2^2\right]
\end{aligned}
,
$$

where the origin is at O. E is Young's modulus, ν is Poisson's ratio, and I is the area moment of inertia of the cross-section about the x_2 axis.

(a) Evaluate the components of the infinitesimal strain tensor.

(b) Verify that the components of the Cauchy stress tensor $\mathbf{T}$ are given by

$$T_{11} = T_{12} = T_{22} = 0, \qquad T_{23} = -\frac{P(1+2\nu)}{4I(1+\nu)} x_1 x_2,\ T_{33} = -\frac{P}{I} x_1(\ell - x_3)$$

$$T_{13} = \frac{P}{8I(1+\nu)}\left[(3+2\nu)\left(a^2 - x_1^2\right) - (1-2\nu)x_2^2\right]$$

(c) Verify that this state of stress satisfies the balance of linear momentum under static conditions, in the absence of body forces.

(d) Verify that this state of stress produces a traction-free lateral boundary.

Hint: The unit outward normal for a circle in the $x_1 - x_2$ plane centered at the origin is given by

$$\mathbf{n} = \frac{x_1}{\sqrt{x_1^2 + x_2^2}}\mathbf{e}_1 + \frac{x_2}{\sqrt{x_1^2 + x_2^2}}\mathbf{e}_2 .$$

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Problem 2

Formulate the statement of balance of angular momentum in integral form and derive its differential form: Assume that a body couple per unit volume, $\mathbf{m}$, acts on the continuum and add this body couple to the integral form of the angular momentum balance as the term

$$\int_V \mathbf{m}\, dV ,$$

where V is the control volume. Show from your analysis that $\mathbf{m} = \mathbb{E} : \mathbf{T}$, or equivalently $m_i = \varepsilon_{ijk} T_{jk}$ in indicial form. *Hint:* You do not need to re-derive the balance of mass or balance of linear momentum.

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Problem 3

In continuum mechanics the entropy inequality is given by

$$-\rho\left(\frac{D\psi}{Dt}+\eta\frac{D\theta}{Dt}\right)+\mathbf{T}:\mathbf{D}-\frac{1}{\theta}\mathbf{q}\cdot\operatorname{grad}\theta\geq 0$$

where ρ is the density, ψ is the specific free energy, η is the specific entropy, θ is the absolute temperature, $\mathbf{T}$ is the Cauchy stress tensor, $\mathbf{D}$ is the rate of deformation tensor, and $\mathbf{q}$ is the heat flux. To model a compressible viscous fluid under isothermal conditions, let ψ, η, $\mathbf{T}$ and $\mathbf{q}$ depend only on $(\rho,\theta,\mathbf{D})$.

(A) Using the chain rule of differentiation, evaluate $D\psi/Dt$ and substitute the result into the entropy inequality. Use appropriate relations (such as the balance of mass) to group like terms together, and specify the constraints imposed on $\psi,\eta,$ and $\mathbf{q}$ by the entropy inequality. What is the resulting reduced form of the entropy inequality? *Hint:* Note that the material time derivative of the rate of deformation tensor, $D\mathbf{D}/Dt$, is a kinematic variable that may vary arbitrarily.

(B) For a Newtonian viscous fluid, the Cauchy stress is generally given by

$$\mathbf{T}=-p\mathbf{I}+\lambda(\operatorname{tr}\mathbf{D})\mathbf{I}+2\mu\mathbf{D}$$

where p is sometimes called the *thermodynamic* pressure; λ and μ are the first and second viscosity coefficients. Using the reduced form of the entropy inequality obtained in part (A), how is p related to ρ and ψ?

(C) Using the reduced form of the entropy inequality obtained in parts (A) and (B), what are the restrictions on the viscosity coefficients λ and μ?

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Problem 1

The constitutive relation for a transversely isotropic linear elastic solid is

$$\mathbf{T}=\lambda_T(\mathrm{tr}\,\mathbf{E})\mathbf{I}+(\lambda_T+\lambda-2\lambda_L)\mathrm{tr}(\mathbf{A}\cdot\mathbf{E})\mathbf{A}+(\lambda_L-\lambda_T)\left[(\mathrm{tr}\,\mathbf{E})\mathbf{A}+\mathrm{tr}(\mathbf{A}\cdot\mathbf{E})\mathbf{I}\right]$$
$$+2\mu_T\mathbf{E}+(\mu-\mu_T)(\mathbf{A}\cdot\mathbf{E}+\mathbf{E}\cdot\mathbf{A})$$

where $\lambda,\mu,\lambda_T,\mu_T,\lambda_L$ are material constants, and $\mathbf{A}=\mathbf{a}\otimes\mathbf{a}$ is a structural tensor corresponding the preferred material orientation $\mathbf{a}$ (a unit vector).

(A) Consider the special case when $\mathbf{a}=\mathbf{e}_3$. Evaluate the matrix $[\mathbf{A}]$ in the Cartesian basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$. Also evaluate $\mathrm{tr}(\mathbf{A}\cdot\mathbf{E})$ and the matrix $[\mathbf{A}\cdot\mathbf{E}+\mathbf{E}\cdot\mathbf{A}]$.

(B) Using the results of (A), evaluate the matrix $[\mathbf{T}]$.

(C) Using the canonical problem of uniaxial loading of a rectangular prismatic bar along the $\mathbf{e}_3$ direction, determine the Young's modulus $E_Y=T_{33}/E_{33}$ and Poisson's ratio $\nu=-E_{11}/E_{33}$ in terms of the material constants $\lambda,\mu,\lambda_T,\mu_T,\lambda_L$.

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Problem 2

(A) Formulate the statement of balance of mass in integral form over a control volume. Using the divergence theorem, derive the differential form of the statement of mass balance.

(B) Formulate the statement of balance of linear momentum in integral form over a control volume. Using the divergence theorem, derive the differential form of the statement of linear momentum balance.

(C) Formulate the statement of balance of energy in integral form over a control volume. Using the divergence theorem, derive the differential form of the statement of energy balance.

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Problem 3

In continuum mechanics the entropy inequality is given by

$$-\rho\left(\frac{D\psi}{Dt}+\eta\frac{D\theta}{Dt}\right)+\mathbf{T}:\mathbf{D}-\frac{1}{\theta}\mathbf{q}\cdot\operatorname{grad}\theta\geq 0$$

where ρ is the density, ψ is the specific free energy, η is the specific entropy, θ is the absolute temperature, $\mathbf{T}$ is the Cauchy stress tensor, $\mathbf{D}$ is the rate of deformation tensor, and $\mathbf{q}$ is the heat flux. To model a compressible viscous fluid under isothermal conditions, let ψ, η, $\mathbf{T}$ and $\mathbf{q}$ depend only on $(\rho,\theta,\mathbf{D})$.

(A) Using the chain rule of differentiation, evaluate $D\psi/Dt$ and substitute the result into the entropy inequality. Use appropriate relations (such as the balance of mass) to group like terms together, and specify the constraints imposed on $\psi,\eta,$ and $\mathbf{q}$ by the entropy inequality. What is the resulting reduced form of the entropy inequality? *Hint:* Note that the material time derivative of the rate of deformation tensor, $D\mathbf{D}/Dt$, is a kinematic variable that may vary arbitrarily.

(B) For a Newtonian viscous fluid, the Cauchy stress is generally given by

$$\mathbf{T}=-p\mathbf{I}+\lambda(\operatorname{tr}\mathbf{D})\mathbf{I}+2\mu\mathbf{D}$$

where p is sometimes called the *thermodynamic* pressure; λ and μ are the first and second viscosity coefficients. Using the reduced form of the entropy inequality obtained in part (A), how is p related to ρ and ψ?

(C) Using the reduced form of the entropy inequality obtained in parts (A) and (B), what are the restrictions on the viscosity coefficients λ and μ?

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Problem 1

The position $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$ at time t of a particle initially at $\mathbf{X}$ is given by

$$x_1 = X_1 - 2X_2^2 t^2, \qquad x_2 = X_2 - X_3 t, \qquad x_3 = X_3$$

(a) Evaluate the particle velocity $\mathbf{v} = \hat{\mathbf{v}}(\mathbf{X}, t)$, i.e., express the velocity in terms of the components of $\mathbf{X}$. $\hat{\mathbf{v}}$ represents the particle velocity in the *material frame*.

(b) Solve for the components of $\mathbf{X}$ in terms of the components of $\mathbf{x}$ and substitute this solution into your result of part (a) to represent the particle velocity as $\mathbf{v} = \tilde{\mathbf{v}}(\mathbf{x}, t)$, i.e., express the velocity in terms of the components of $\mathbf{x}$. $\tilde{\mathbf{v}}$ represents the particle velocity in the *spatial frame*.

(c) Evaluate the particle acceleration in the material frame, $\mathbf{a} = \hat{\mathbf{a}}(\mathbf{X}, t)$, by differentiating $\hat{\mathbf{v}}(\mathbf{X}, t)$. Also evaluate the acceleration in the spatial frame, $\mathbf{a} = \tilde{\mathbf{a}}(\mathbf{x}, t)$ by using the material derivative in the spatial frame, $\tilde{\mathbf{a}} = D\tilde{\mathbf{v}}/Dt$ (show details). Show that the results for $\hat{\mathbf{a}}$ and $\tilde{\mathbf{a}}$ are equivalent.

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Problem 2

In linear elasticity, the constitutive relation between the stress and the strain is given by

$$T_{ij} = \frac{\partial U}{\partial E_{ij}}$$

where T_{ij} are the components of the Cauchy stress tensor $\mathbf{T}$, E_{ij} are the components of the infinitesimal strain tensor $\mathbf{E}$, and U is the strain energy density. In general, U may be expressed in terms of the invariants of the strain tensor. For example, $U = \hat{U}(I_1, I_2)$ where

$$I_1 = \operatorname{tr}\mathbf{E} = E_{kk}, \qquad I_2 = \frac{1}{2}\left[(\operatorname{tr}\mathbf{E})^2 - \operatorname{tr}\left(\mathbf{E}^2\right)\right] = \frac{1}{2}\left(E_{kk}E_{ll} - E_{kl}E_{kl}\right)$$

(a) What should be the form of $\hat{U}(I_1, I_2)$ to recover the classical Hooke's law for a linear isotropic elastic solid? (Formulate an expression for the dependence of $\hat{U}$ on I_1 and I_2 and use the symbols λ and μ for the Lamé constants.)

(b) Since I_2 depends on I_1, it is also possible to define an alternative set of strain invariants as

$$J_1 = \operatorname{tr}\mathbf{E} = E_{kk}, \qquad J_2 = \operatorname{tr}\left(\mathbf{E}^2\right) = E_{kl}E_{kl}$$

Find the equivalent form of $U = \tilde{U}(J_1, J_2)$ that produces the classical Hooke's law for a linear isotropic elastic solid. (Formulate the expression for the dependence of $\tilde{U}$ on J_1 and J_2 and use the shear modulus μ and bulk modulus κ for the material constants.)

(c) When a material is incompressible, its bulk modulus becomes infinite ($\kappa \to \infty$) and its dilatation reduces to zero ($\operatorname{tr}\mathbf{E} \to 0$). Flory (1961) proposed to describe nearly-incompressible solids by separating the strain energy density into a term depending exclusively on the dilatation and a term independent of the dilatation,

$$\tilde{U}(J_1, J_2) = \tilde{U}_1(J_1) + \tilde{U}_2(J_2)$$

Show that your result from part (b) may be written in this form. Then, letting

$$\lim_{\substack{\kappa \to \infty \\ J_1 \to 0}} \frac{d\tilde{U}_1}{dJ_1} = -p$$

write the resulting expression for the stress $\mathbf{T}$ for an incompressible linear isotropic elastic solid.

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Problem 3

For any stress state $\mathbf{T}$, we define the deviatoric stress $\mathbf{S}$ to be

$$\mathbf{S} = \mathbf{T} - \left(\frac{T_{kk}}{3}\right)\mathbf{I}$$

where T_{kk} is the first invariant of the stress tensor $\mathbf{T}$.

(a) Show that the first invariant of the deviatoric stress vanishes.
(b) Show that the principal direction of the stress and deviatoric stress coincide.
(c) Find a relation between the principal values of the stress and the deviatoric stress.
(d) Given the stress tensor

$$[\mathbf{T}] = \begin{bmatrix} -p & \tau & 0 \\ \tau & -p & 0 \\ 0 & 0 & -p \end{bmatrix}$$

evaluate $\mathbf{S}$. Also find the principal values and principal directions of $\mathbf{T}$.

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Problem 1:

Let $\mathbf{Q}$ be an orthogonal transformation that transforms the basis $\{\mathbf{e}_i\}$ into $\{\mathbf{e}'_i\}$ according to

$$\mathbf{e}'_i = \mathbf{Q}\cdot\mathbf{e}_i \tag{1}$$

a) Any vector $\mathbf{v}$ may be represented in terms of its components as

$$\mathbf{v} = v_i\mathbf{e}_i \tag{2}$$

in $\{\mathbf{e}_i\}$ and as

$$\mathbf{v} = v'_i\mathbf{e}'_i \tag{3}$$

in $\{\mathbf{e}'_i\}$. Find the general relation between v'_i and v_i.

b) Any second-order tensor $\mathbf{T}$ may be similarly represented as

$$\mathbf{T} = T_{ij}\mathbf{e}_i\otimes\mathbf{e}_j \tag{4}$$

in $\{\mathbf{e}_i\}$ and

$$\mathbf{T} = T'_{ij}\mathbf{e}'_i\otimes\mathbf{e}'_j \tag{5}$$

in $\{\mathbf{e}'_i\}$. Find the general relation between T'_{ij} and T_{ij}.

c) Any fourth-order tensor $\mathbf{C}$ may be represented as

$$\mathbf{C} = C_{ijkl}\mathbf{e}_i\otimes\mathbf{e}_j\otimes\mathbf{e}_k\otimes\mathbf{e}_l \tag{6}$$

in $\{\mathbf{e}_i\}$ and

$$\mathbf{C} = C'_{ijkl}\mathbf{e}'_i\otimes\mathbf{e}'_j\otimes\mathbf{e}'_k\otimes\mathbf{e}'_l \tag{7}$$

in $\{\mathbf{e}'_i\}$. Find the general relation between C'_{ijkl} and C_{ijkl}.

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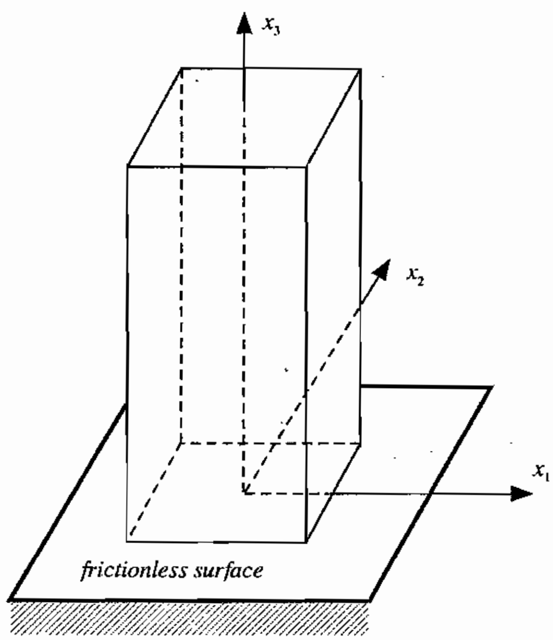
Problem 2:

Under static conditions, the equation of conservation of linear momentum is given by

$$\operatorname{div}\mathbf{T} + \rho\mathbf{b} = \mathbf{0} \tag{1}$$

where $\mathbf{T}$ is the Cauchy stress tensor, ρ is the density of the material, and $\mathbf{b}$ is a body force per unit mass.

a) Assuming that the body force represents gravity, with $\mathbf{b} = -g\mathbf{e}_3$, write the equation of linear momentum in terms in component form.



b) Consider a prismatic rectangular block resting on the frictionless plane whose unit outward normal is $\mathbf{e}_3$, under the action of its own weight. Specify the boundary conditions on all the faces of the prismatic bar for this problem. Solve for the stresses from the knowledge of these boundary conditions, and using Eq.(1).

c) Let the prismatic bar be modeled as a linear isotropic elastic solid, with $\mathbf{T} = \lambda\,\mathrm{tr}(\mathbf{E})\mathbf{I} + 2\mu\mathbf{E}$, where λ and μ are the Lamé constants and $\mathbf{E}$ is the infinitesimal strain tensor. Solve for the components of the strain tensor, given the boundary conditions and governing equations for this problem.

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Problem 3:

The constitutive relation for a linear isotropic elastic solid is given by

$$\mathbf{T} = \lambda(\operatorname{tr}\mathbf{E})\mathbf{I} + 2\mu\mathbf{E} \tag{2}$$

where $\mathbf{T}$ is the Cauchy stress tensor and $\mathbf{E}$ is the infinitesimal strain tensor.

a) Using indicial notation, write the general expression for the components C_{ijkl} of the elasticity tensor $\mathbf{C}$. Show how you obtained your result.

b) Write the 6×6 matrix form of $\mathbf{C}$ in the Cartesian basis $\{\mathbf{e}_i\}$ such that

$$\begin{bmatrix} T_{11} \\ T_{22} \\ T_{33} \\ T_{23} \\ T_{13} \\ T_{12} \end{bmatrix} = [\mathbf{C}] \begin{bmatrix} E_{11} \\ E_{22} \\ E_{33} \\ 2E_{23} \\ 2E_{13} \\ 2E_{12} \end{bmatrix} \tag{3}$$

1 Problem 1

If a thin plate is loaded by forces applied on its boundary, parallel to the plane of the plate and distributed uniformly over the thickness, we call this a state of plane stress. Then, the stress tensor is given by

$$[\mathbf{T}] = \begin{bmatrix} T_{11}(x_1,x_2) & T_{12}(x_1,x_2) & 0 \\ T_{12}(x_1,x_2) & T_{22}(x_1,x_2) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

1. Write the equations of static equilibrium for plane stress, in component form, in the absence of body forces.

2. We would like to solve for the stresses in the cantilevered beam shown in the figure.

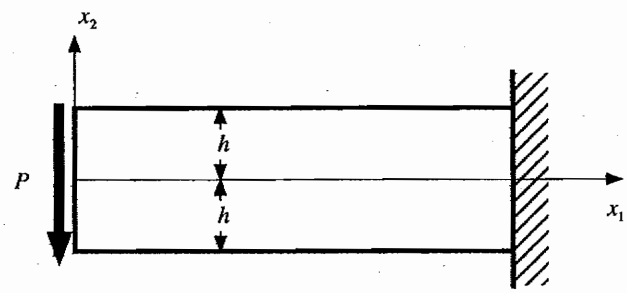

Figure 1.1: Cantileve beam, with end load P.

Assume that the stresses are generally given by the polynomial forms

$$T_{11} = a_{00} + a_{10}x_1 + a_{01}x_2 + a_{11}x_1x_2 + a_{20}x_1^2 + a_{02}x_2^2$$
$$T_{22} = b_{00} + b_{10}x_1 + b_{01}x_2 + b_{11}x_1x_2 + b_{20}x_1^2 + b_{02}x_2^2$$
$$T_{12} = c_{00} + c_{10}x_1 + c_{01}x_2 + c_{11}x_1x_2 + c_{20}x_1^2 + c_{02}x_2^2$$

where a_{ij}, b_{ij} and c_{ij} are constant coefficients, to be determined. What are the constraints on these coefficients imposed by the equations of static equilibrium?

3. What are the constraints on these coefficients resulting from the traction boundary conditions on the top and bottom faces, and the left face of the cantilever beam?

4. Using the results of parts 2 and 3, solve for all the unknown coefficients and present the final result for T_{11}, T_{22} and T_{12}.

2 Problem 2

1. Derive the equation of conservation of linear momentum for a continuum from basic principles.

2. Derive the equation of conservation of angular momentum for a continuum from basic principles.

3 Problem 3

The constitutive relation for a transversely isotropic elastic solid under infinitesimal strain is given by

$$\mathbf{T} = \lambda_T (\mathrm{tr}\,\mathbf{E})\,\mathbf{I} + (\lambda_T + \lambda - 2\lambda_L)(\mathrm{tr}\,\mathbf{AE})\,\mathbf{A} + (\lambda_L - \lambda_T)\left[(\mathrm{tr}\,\mathbf{E})\,\mathbf{A} + (\mathrm{tr}\,\mathbf{AE})\,\mathbf{I}\right] + 2\mu_T\mathbf{E} + (\mu - \mu_T)(\mathbf{AE} + \mathbf{EA})$$

where $\mathbf{E}$ is the infinitesimal strain tensor and $\mathbf{A} = \mathbf{a} \otimes \mathbf{a}$ is a texture tensor, with $\mathbf{a}$ representing the unit vector normal to the transverse plane of isotropy; λ, λ_T, λ_L, μ, and μ_T are the material constants for this type of solid.

1. Assume that $\mathbf{a} = \mathbf{e}_3$ (i.e., $[\mathbf{a}] = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$). Evaluate the matrix $[\mathbf{A}]$ in this cartesian coordinate system, as well as $[\mathbf{AE} + \mathbf{EA}]$ and $\mathrm{tr}\,\mathbf{AE}$.

2. Using these results, write the general expression for each of the components of $[\mathbf{T}]$ in terms of the components of $[\mathbf{E}]$.

3. Consider uniaxial loading of a prismatic bar of rectangular cross-section, with the bar aligned and loaded along $\mathbf{e}_3$.

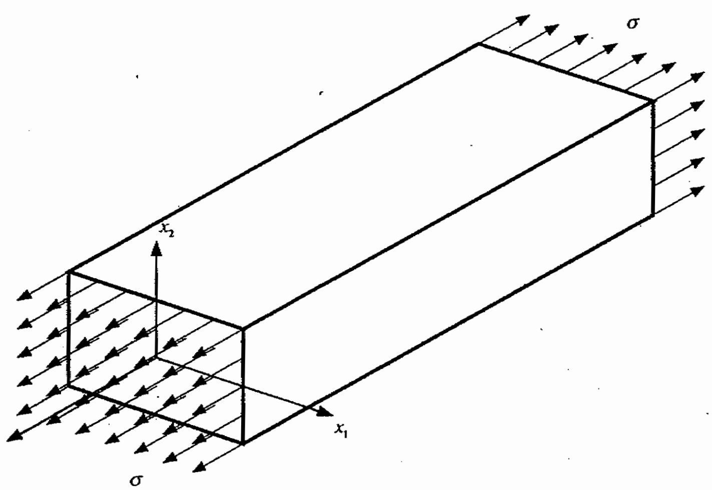

Figure 3.1: Uniaxial loading of a prismatic bar.

What are the traction boundary conditions on the boundaries of this bar?

4. Given your results for the previous sections, solve for all the strain components in the bar.

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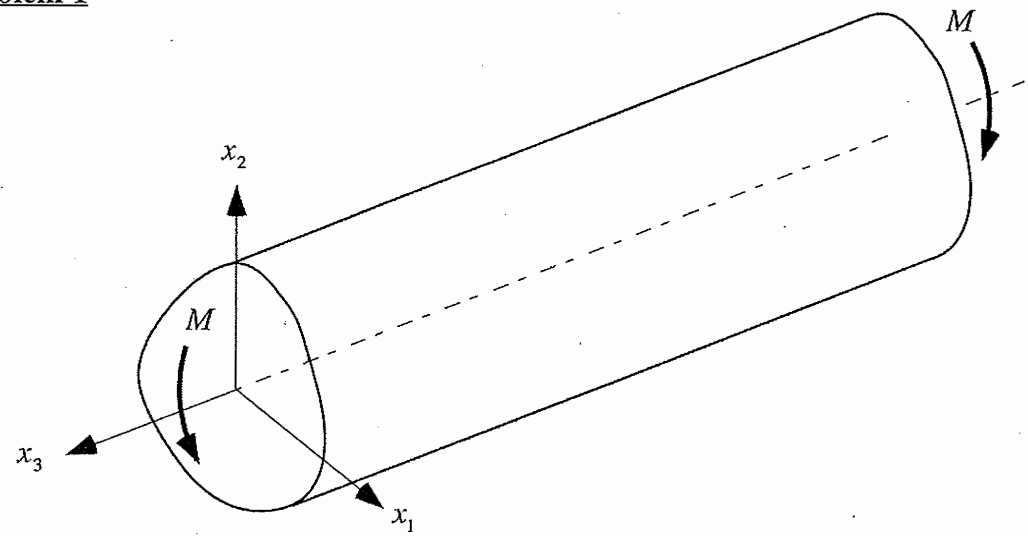
Problem 1



In the problem of pure bending of a prismatic beam of arbitrary cross-section about the x_1 – axis, the stress tensor is given by

$$[\mathbf{T}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & T_{33} \end{bmatrix}$$

where

$$T_{33}(x_1, x_2) = \alpha + \beta x_1 + \gamma x_2$$

a) Show that this stress tensor satisfies the equations of stress equilibrium.
b) Determine the traction vector on all the boundaries.
c) What is the strain tensor for this problem, assuming that the beam is a linear elastic isotropic solid?
d) What should the values of α , β and γ be in order to satisfy the boundary conditions for this problem?

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Problem 2

Find the solution for the strains and displacements in a spherical shell of inner radius a and outer radius b, subjected to an internal pressure p_i and external pressure p_e. The shell is linear isotropic elastic.

Hints:

In the absence of body forces the equations of static equilibrium in spherical coordinates, are given by

$$\frac{\partial T_{rr}}{\partial r}+\frac{1}{r\sin\phi}\frac{\partial T_{r\theta}}{\partial \theta}+\frac{1}{r}\frac{\partial T_{r\phi}}{\partial \phi}+\frac{T_{r\phi}}{r}\cot\phi+\frac{2T_{rr}-T_{\theta\theta}-T_{\phi\phi}}{r}=0$$

$$\frac{\partial T_{r\theta}}{\partial r}+\frac{1}{r\sin\phi}\frac{\partial T_{\theta\theta}}{\partial \theta}+\frac{1}{r}\frac{\partial T_{\theta\phi}}{\partial \phi}+\frac{2T_{\theta\phi}}{r}\cot\phi+\frac{3T_{r\theta}}{r}=0$$

$$\frac{\partial T_{r\phi}}{\partial r}+\frac{1}{r\sin\phi}\frac{\partial T_{\theta\phi}}{\partial \theta}+\frac{1}{r}\frac{\partial T_{\phi\phi}}{\partial \phi}+\frac{T_{\phi\phi}-T_{\theta\theta}}{r}\cot\phi+\frac{3T_{r\phi}}{r}=0$$

where T_{ij} $(i,j=r,\theta,\phi)$ are the components of the stress tensor $\mathbf{T}$.

In spherical coordinates the infinitesimal strain tensor is given by

$$E_{rr}=\frac{\partial u_r}{\partial r} \qquad E_{\theta\phi}=\frac{1}{2}\left[\frac{1}{r}\frac{\partial u_\theta}{\partial \phi}+\frac{1}{r\sin\phi}\left(\frac{\partial u_\phi}{\partial \theta}-u_\theta\cos\phi\right)\right]$$

$$E_{\theta\theta}=\frac{1}{r\sin\phi}\frac{\partial u_\theta}{\partial \theta}+\frac{u_r}{r}+\frac{\cot\phi}{r}u_\phi \qquad E_{r\phi}=\frac{1}{2}\left[\frac{1}{r}\left(\frac{\partial u_r}{\partial \phi}-u_\phi\right)+\frac{\partial u_\phi}{\partial r}\right]$$

$$E_{\phi\phi}=\frac{1}{r}\left(\frac{\partial u_\phi}{\partial \phi}+u_r\right) \qquad E_{r\theta}=\frac{1}{2}\left[\frac{1}{r\sin\phi}\left(\frac{\partial u_r}{\partial \theta}-u_\theta\sin\phi\right)+\frac{\partial u_\theta}{\partial r}\right]$$

Problem 3

The energy equation for a continuum is given by

$$\rho \frac{D\varepsilon}{Dt} = \mathbf{T} : \mathbf{D} - \operatorname{div}\mathbf{q} + \rho r$$

where ρ is the density, ε is the internal energy per unit mass, $\mathbf{T}$ is the stress tensor, $\mathbf{D} = \left(\operatorname{grad}\mathbf{v} + \operatorname{grad}^T \mathbf{v}\right)/2$ is the rate of deformation tensor (where $\mathbf{v}$ is the velocity), $\mathbf{q}$ is the heat flux and r is the heat supply from external sources. The entropy inequality is given by

$$\rho \frac{D\eta}{Dt} + \operatorname{div}\left(\frac{\mathbf{q}}{\theta}\right) - \frac{\rho r}{\theta} \geq 0$$

where η is the entropy per unit mass and θ is the absolute temperature.

a) Given that $\psi = \varepsilon - \theta\eta$ is the Helmholtz free energy per unit mass, derive an expression for the entropy inequality in terms of $D\psi/Dt$ instead of $D\eta/Dt$. Make use of other standard conservation laws if necessary.
b) Give a definition of free energy.
c) What does free energy represent in an elastic solid?
d) Let $\psi = \psi(\mathbf{E}, \theta)$, where $\mathbf{E} = \left(\operatorname{grad}\mathbf{u} + \operatorname{grad}^T \mathbf{u}\right)/2$ is the infinitesimal strain tensor. Using the chain rule of differentiation, substitute this result into the expression obtained in (a) and find a relation between $\mathbf{T}$ and the free energy.

ELASTICITY

Doctoral Qualifying Examination, January 2006
Mechanical Engineering, Columbia University

Problem 1:

The constitutive relation for the stress in a linear isotropic elastic solid is given by

$$\sigma = \lambda(\operatorname{tr}\mathbf{E})\mathbf{I} + 2\mu\mathbf{E}$$

where $\mathbf{E}$ is the infinitesimal strain tensor.

a) What is the 6×6 matrix of the elasticity tensor?

b) Derive the restrictions on the material properties λ and μ from basic principles.

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Doctoral Qualifying Examination, January 2006
Mechanical Engineering, Columbia University

Problem 2:

Consider a rectangular prismatic bar of length l, whose axis is along $\mathbf{e}_3$ in the Cartesian basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. The bar is subjected to a gravitational body force $\mathbf{b} = -g\mathbf{e}_3$ and is suitably constrained at the top end so that static equilibrium can be satisfied. The bottom end is free.

a. Formulate the equations of static equilibrium for this problem.

b. Formulate the complete set of boundary conditions and explain any assumptions.

c. Solve for all the components of the stress tensor, assuming that the material is homogeneous.

d. Solve for all the components of the strain tensor, assuming that the material is linear isotropic elastic.

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Doctoral Qualifying Examination, January 2006
Mechanical Engineering, Columbia University

Problem 3:

According to the second law of thermodynamics, for processes under constant temperature,

$$-\rho \frac{D\psi}{Dt} + \boldsymbol{\sigma} : \mathbf{D} \geq 0$$

where ρ is the density, ψ is the specific Helmholtz free energy, $\boldsymbol{\sigma}$ is the stress tensor, and $\mathbf{D}$ is the rate of deformation tensor. Assuming that ψ is only a function of the right Cauchy-Green strain tensor $\mathbf{C}$, $\psi = \psi(\mathbf{C})$, derive the constitutive restriction on $\boldsymbol{\sigma}$ which satisfies the second law.

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ELASTICITY **Doctoral Qualifying Examination, January 2005**
Mechanical Engineering, Columbia University

Problem 1

Consider the velocity field of a continuum whose components are given by

$$v_i = \frac{x_i}{1+t}$$

where x_i are the Cartesian coordinates of the position vector in the current configuration and t is time.

a) Find the density of a material particle in this continuum, as a function of time.
b) Find the acceleration of a material particle in this continuum.
c) Find the pathline $x_i = x_i(\mathbf{X},t)$, where $\mathbf{X}$ is the position vector in the reference configuration.
d) From the results of part c) determine the deformation gradient for this motion.
e) From the results of part d) determine the Lagrangian strain tensor for this motion.

(305)

ELASTICITY **Doctoral Qualifying Examination, January 2005**
Mechanical Engineering, Columbia University

Problem 2

The governing partial differential equation for torsion of a prismatic bar is given by

$$\frac{\partial^2 \varphi}{\partial x_1^2}+\frac{\partial^2 \varphi}{\partial x_2^2}=0$$

where $\varphi(x_1, x_2)$ is the warping function. This PDE needs to be solved over the domain of the bar cross-section, subject to the boundary condition

$$\frac{\partial \varphi}{\partial x_1} n_1+\frac{\partial \varphi}{\partial x_2} n_2=x_2 n_1-x_1 n_2$$

on the boundary curve C, where n_1, n_2 are the components of the outward unit normal on the boundary. The torque-twist relationship is given by

$$M=\mu \theta\left[I_p+\int_A\left(x_1 \frac{\partial \varphi}{\partial x_2}-x_2 \frac{\partial \varphi}{\partial x_1}\right) dA\right]$$

where M is the torque (moment) on the cross-section A, I_p is the polar moment of inertia of the cross-section, and θ is the angle of twist per unit length.

a) Find the solution for $\varphi(x_1, x_2)$ for a circular cross-section of radius a, then derive the torque-twist relationship for that case.
b) Find the solution for $\varphi(x_1, x_2)$ for an elliptical cross-section of semi-major and semi-minor radii a and b, then derive the torque-twist relationship for that case.

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Problem 3

For a linear isotropic elastic material undergoing infinitesimal strains, the momentum equations in cylindrical coordinates (r,θ,z) are given by

$$\frac{\partial\sigma_{rr}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{r\theta}}{\partial\theta}+\frac{\sigma_{rr}-\sigma_{\theta\theta}}{r}+\frac{\partial\sigma_{rz}}{\partial z}=0$$

$$\frac{\partial\sigma_{\theta r}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial\theta}+\frac{\sigma_{r\theta}+\sigma_{\theta r}}{r}+\frac{\partial\sigma_{\theta z}}{\partial z}=0$$

$$\frac{\partial\sigma_{zr}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{z\theta}}{\partial\theta}+\frac{\sigma_{zr}}{r}+\frac{\partial\sigma_{zz}}{\partial z}=0$$

a) Reduce these equations to the case of axisymmetric conditions.

b) For axisymmetric conditions the infinitesimal strain tensor is given by

$$[\mathbf{E}]=\begin{bmatrix}\dfrac{\partial u_r}{\partial r} & \dfrac{1}{2}\left(\dfrac{\partial u_\theta}{\partial r}-\dfrac{u_\theta}{r}\right) & \dfrac{1}{2}\left(\dfrac{\partial u_r}{\partial z}+\dfrac{\partial u_z}{\partial r}\right)\\ \dfrac{1}{2}\left(\dfrac{\partial u_\theta}{\partial r}-\dfrac{u_\theta}{r}\right) & \dfrac{u_r}{r} & \dfrac{1}{2}\dfrac{\partial u_\theta}{\partial z}\\ \dfrac{1}{2}\left(\dfrac{\partial u_r}{\partial z}+\dfrac{\partial u_z}{\partial r}\right) & \dfrac{1}{2}\dfrac{\partial u_\theta}{\partial z} & \dfrac{\partial u_z}{\partial z}\end{bmatrix}$$

where u_r, u_θ, u_z are the radial, circumferential and axial components of the displacement vector. Determine the components of the stress tensor in the case of a linear isotropic elastic material.

c) Using the results of parts a) and b), evaluate the momentum equations for the case of axisymmetry.

d) Consider a hollow cylindrical tube of inner radius a and outer radius b, subjected to an axial normal traction σ_0 along the front and back annular faces. Formulate the complete set of boundary conditions for this problem in terms of the displacement components.

e) Find the solution for u_r, u_θ, u_z for the problem described in part d), using the governing equations derived in part c). Assume that there is no rigid body motion.

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ELASTICITY

Doctoral Qualifying Examination, January 2004
Mechanical Engineering, Columbia University

1. If x_1x_2 plane is the only plane of material symmetry for an elastic medium, which component(s) of the elasticity tensor C_{ijkl} is (are) zero? Show how you arrive at the answer for one of those C_{ijkl}.

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2. Derive the following Navier equations for a linear isotropic elastic medium in equilibrium

$$(\lambda+\mu)\frac{\partial e}{\partial x_i}+\mu\frac{\partial^2 u_i}{\partial x_j \partial x_j}+\rho_o B_i=0.$$

Where $e=E_{kk}$ is the dilatation, B_i is the body force per unit mass.

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3. When a non-circular cylindrical bar, with its length in the e_1 direction, is in equilibrium under torsion, each of its cross sections becomes warped, but otherwise rotates like a rigid disc.

(a) Express the above statement in mathematical form for small twisting angles
(b) Show that only circular cross section does not warp under torsion.

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4. A circular cylindrical bar (radius r_o), with its length in the e_1 direction, is under the action of equal and opposite parabolic distribution of tensile surface traction $t=\pm C\left(r_o^2-r^2\right)e_1$ at its end faces.

(a) Explain why the distribution inside the bar is in general not given by

$$[T]=\begin{bmatrix} C\left(r_o^2-r^2\right) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(b) If the bar is very long, describe the state of stress inside the bar.
(c) If the bar is very short, describe the state of stress inside the bar.

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$$\boldsymbol{T}\boldsymbol{e}_i = T_{mi}\boldsymbol{e}_m, \quad T_{ij} = \boldsymbol{e}_i \cdot \boldsymbol{T}\boldsymbol{e}_j$$

$$T_{ij}' = Q_{mi}Q_{nj}T_{mn}, \quad C_{ijkl}' = Q_{mi}Q_{nj}Q_{rk}Q_{sl}C_{mnrs}$$

$$[\nabla u] = \begin{bmatrix} \dfrac{\partial u_1}{\partial X_1} & \dfrac{\partial u_1}{\partial X_2} & \dfrac{\partial u_1}{\partial X_3} \\ \dfrac{\partial u_2}{\partial X_1} & \dfrac{\partial u_2}{\partial X_2} & \dfrac{\partial u_2}{\partial X_3} \\ \dfrac{\partial u_3}{\partial X_1} & \dfrac{\partial u_3}{\partial X_2} & \dfrac{\partial u_3}{\partial X_3} \end{bmatrix}, \quad E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right)$$

$$\frac{\partial^2 E_{11}}{\partial X_2^2} + \frac{\partial^2 E_{22}}{\partial X_1^2} = 2\frac{\partial^2 E_{12}}{\partial X_1 \partial X_2}, \quad \frac{\partial^2 E_{11}}{\partial X_2 \partial X_3} = \frac{\partial}{\partial X_1}\left(-\frac{\partial E_{23}}{\partial X_1} + \frac{\partial E_{31}}{\partial X_2} + \frac{\partial E_{12}}{\partial X_3}\right)$$

$$\boldsymbol{t} = \boldsymbol{T}\boldsymbol{n}, \quad t_i = T_{ij}n_j$$

$$\frac{\partial T_{ij}}{\partial x_j} + \rho_o B_i = 0$$

$$\boldsymbol{T} = \lambda E_{kk}\boldsymbol{I} + 2\mu\boldsymbol{E}, \quad \boldsymbol{E} = \frac{1}{E_Y}\left[(1+\nu)\boldsymbol{T} - \nu T_{kk}\boldsymbol{I}\right],$$

$$T_{11} = \frac{\partial^2 \varphi}{\partial x_2^2}, \quad T_{22} = \frac{\partial^2 \varphi}{\partial x_1^2}, \quad T_{12} = -\frac{\partial^2 \varphi}{\partial x_1 \partial x_2}$$

$$\frac{\partial^4 \varphi}{\partial x_1^4} + 2\frac{\partial^4 \varphi}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4 \varphi}{\partial x_2^4} = 0$$