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Doctoral Qualifying Examination, May 2016

Mechanical Engineering, Columbia University

DYNAMICS

Problem 1

A ball of mass m is attached to a thin rigid rod of mass M, length L, and mass moment of inertia I ($=\frac{ML^2}{3}$) about the pivot at the wall as shown in the figure below. The ball-rod assembly can rotate in the vertical plane under the action of gravity and a linear spring, of stiffness K, attached at its center. The pivot is frictionless. The spring as shown is at its unstretched position at $\theta = 0$, the angle that the rod makes with the horizontal.

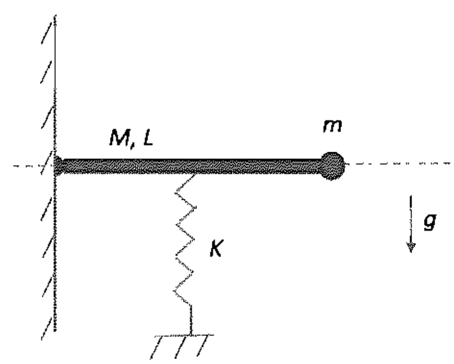

Figure 1 – Ball-rod Assembly Rotating in Vertical Plane

Draw the assembly at an arbitrary θ above the horizontal and: **1 pt**

a. Derive the potential energy of this dynamic system **2 pts**

Hint: may assume spring stretch as $\frac{L}{2} \sin\theta$

b. Derive the kinetic energy of the system **2 pts**

c. Derive its nonlinear second order differential equation of motion in θ. **4 pts**

d. What term would you add to this equation had the pivot not been well lubricated? **1 pt**

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DYNAMICS

Problem 2

A slender rod of mass m and length l is centrally mounted on shaft A-A' about which it is rotating with a constant spin $\dot{\emptyset}$. Simultaneously, the yoke is forced to rotate with a constant angular velocity ω_0 due to a moment $\bar{M}$ as shown. ($I_z = \frac{1}{12}ml^2$, rod's mass moment of inertia.)

a. Center the coordinate system attached to the rigid body (rod) at O in a convenient orientation, where the x-axis is along the pointed direction shown Figure 2. Draw a clear schematic. **1pt**

b. Find $\bar{M}$ required to maintain the constant speed ω_0 in terms of geometry, angle $\emptyset$, the spin, and the constant speed. **5 pts**

c. If now a torque T_z is also applied about the z-axis to make the rod have a constant angular acceleration $\ddot{\emptyset}$, express T_z in terms of geometry, $\emptyset$, $\ddot{\emptyset}$, and constant speed ω_0. **3 pts**

d. Explain in words how is $\bar{M}$ affected by the presence of $\ddot{\emptyset}$. **1 pt**

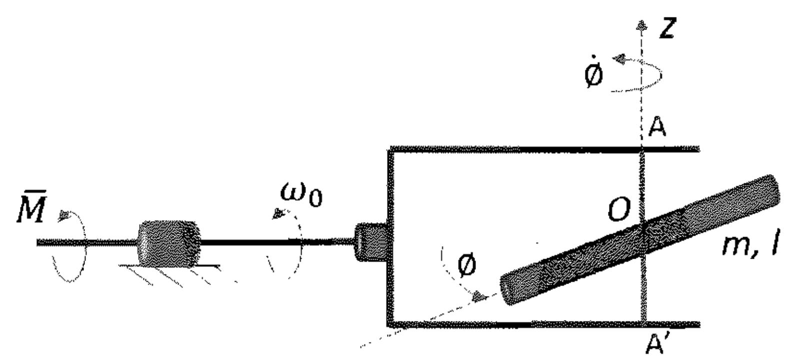

Figure 2 – Spinning Slender Rod in Rotating Yoke

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DYNAMICS

Problem 3

A spacecraft is launched into space spinning about its z-axis that is also its axis of symmetry. It possesses two planes of symmetry, as such the inertia matrix about its center of mass is diagonal with $I_x = I_y$. An approximate rendition is shown in Figure 3.

a. Write the modified Newton-Euler equations appropriate for this moment-free problem. **3 pts**

b. Show that the spin ω_z, the z-component of the angular velocity, is constant. **2 pts**

c. Prove that any rotation about the x- and/or y- axes is sinusoidal; and **4 pts**

d. what is the period of such a wobble (or precession) in terms of geometry and spin? **1 pt**

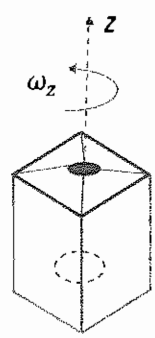

Figure 3 – Spacecraft in Free Rotation

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DYNAMICS

Problem 4

Consider, as shown in Figure 4, a rigid bar of mass m and length l rotating in the vertical plane about frictionless hinge O under the action of gravity. Take the mass moment of inertia of the bar about O as I. Also consider the one d.o.f. θ as the generalized coordinate.

a. Obtain the Lagrangian, $\mathcal{L}$ **2 pts**

b. Define the generalized momentum as $p = \frac{\partial \mathcal{L}}{\partial \dot{\theta}}$, and the Hamiltonian as $\mathcal{H} = \sum_i p_i q_i - \mathcal{L}$, where q_i is the ith generalized coordinate. Apply the Hamilton's Canonical equations to obtain two coupled first order differential equations of motion of this nonlinear dynamic system. **4 pts**

c. Obtain the equilibrium point, and explain why it makes sense. **1 pt**

d. Obtain the $\dot{x} = Ax$ equation so to allow you to study the stability around the equilibrium point, knowing that the state vector is $x = \begin{Bmatrix} \theta \\ p \end{Bmatrix}$. Do not actually do the stability analysis. **3 pts**

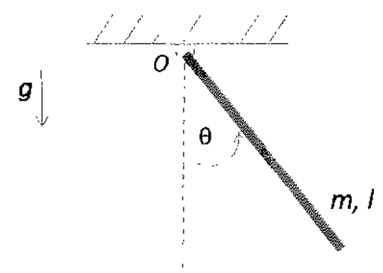

Figure 4 – Rigid Bar Rotating in Vertical Plane

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Mechanical Engineering, Columbia University

DYNAMICS

Problem 1 (Newton-Euler)

A cylinder of known mass and geometry is spinning with constant angular velocity ω_2. Its bearings rest on identical linear springs. See Figure 1. The whole assembly sits in an aircraft as part of its navigational system. Consider that the aircraft makes a turn so the cylinder rotates about *Oy* with a constant velocity ω_1.

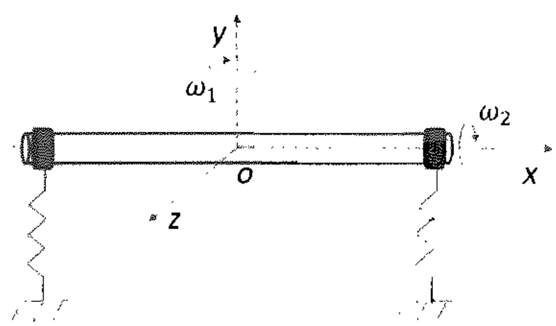

Figure 1

Assume that the rotating frame *Oxyz* is at the center of mass of the cylinder, and that all principal moments of inertia are known.

- **a.** Find the angular momentum of the cylinder about *O* — **2 pts**
- **b.** What is the gyroscopic moment equal to? — **3 pts**
- **c.** Find the reactions at the spring supports (assume no moment reactions). — **5 pts**

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DYNAMICS

Problem 2 (Lagrange)

A pendulum of mass m is attached to a moving cart of mass M via a hinge as shown in Figure 2. The rod is of length l and has negligible mass.

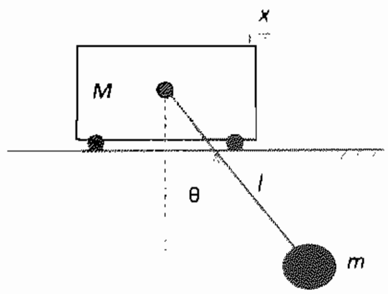

Figure 2

a. Find the equations of motion if friction is neglected **5 pts**

b. Find the equations of motion assuming the hinge is rusty and its viscous damping coefficient is b. Use the Rayleigh Dissipation Function $\mathcal{F} = \frac{1}{2} b\dot{\theta}^2$. **5 pts**

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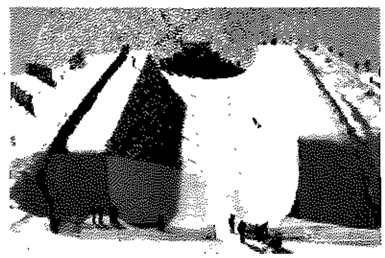
DYNAMICS

Problem 3 (Lagrange, D'Alembert)

A snowboarder of mass m skids down a half-pipe as shown in the figures below.

Snowboarder



Half-pipe

Assume that the snowboarder always stays on the surface of the half-pipe and skids in a 2D plane. Consider the half-pipe as a cylinder whose cross section is an ellipse with major and minor radii a and b, and hence the equation of a half-pipe is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

a. Write the equations of motion of the snowboarder via a Lagrange or a D'Alembert approach **6 pts**

b. Find the normal force between the snowboarder and the half-pipe by using a nonholonomic form of a Lagrange formulation. (Do not actually solve for the constraint force just set up the system of equations and identify the unknowns). **4 pts**

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DYNAMICS

Problem 4.1 (Stability)

A block of mass m is attached to a dashpot of damping coefficient c, and to a nonlinear softening spring of coefficient k, is acted upon by a force $F(t)$. The block's equation of motion is: $m\ddot{x} + c\dot{x} + k\left(x - \frac{x^3}{6}\right) = F(t)$, where x is the position of the block.

- **a.** Describe the steps to obtain the equilibrium points of this dynamic system **2 pts**
- **b.** The linearized perturbation equation is $\dot{\Delta x} = A\Delta x + B\Delta u$, where x is the state, u the input, A the Jacobian matrix, and B the input matrix. *Briefly* describe the steps to obtain A and study it in order to analyze the stability of this nonlinear dynamic system. **3 pts**

Problem 4.2 (Hamilton)

Consider a simple pendulum of mass m and length l in the vertical plane (no frictional effects). Take as generalized coordinate θ, the angular displacement of the massless attaching rod relative to the vertical, and g as the acceleration of gravity.

- **a.** Prove that the Lagrangian $\mathcal{L} = \frac{1}{2}m(l\dot{\theta})^2 - mgl(1 - \cos\theta)$ **2 pts**
- **b.** Obtain the general momentum coordinate p **1 pt**
- **c.** Obtain the Hamiltonian $\mathcal{H}$ **2 pts**
- **d.** *Extra credit*: What is the name of the mathematical transform that takes $\mathcal{L}$ unto $\mathcal{H}$? **0.5 pt**

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DYNAMICS AND VIBRATIONS

Problem 1

A bead of mass m slides smoothly down a parabolic wire, $y = x^2$, under the action of gravity, g, as shown in Figure 1.

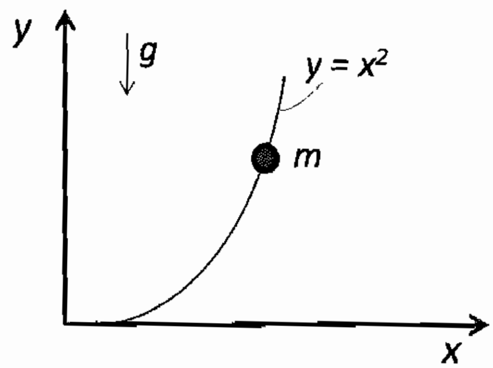

Figure 1

a. Simply write the constraint equation as $f(x,y) = 0$. Obtain df, the differential of f, and express it as a_1dx+a_2dy. Explain why this constraint is holonomic.

b. Assume the bead is at x and y and that its velocity components are $\dot{x}$ and $\dot{y}$. We are interested in solving for its x and y motions and for the constraint force. As such, adding $a_i\lambda$, $i=1,2$, where λ is a Lagrange multiplier, to the holonomic Lagrange equations (right-hand sides) gives rise to the nonholonomic form of the Lagrange equations. Thus:

1. Obtain the two nonholonomic Lagrange equations of motion.
2. We then have 2 equations with 3 unknowns. What are these unknowns?
3. Differentiate the constraint equation to obtain the third differential equation.
4. What does λ represent?

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<u>**DYNAMICS AND VIBRATIONS**</u>

Problem 2

A disk of mass m and radius R is attached to the shaft of a motor that is spinning with constant angular velocity ω_2. Neglect the inertia of the shaft. The motor sits rigidly on a turntable that is rotating with angular velocity ω_1. See Figure 2.

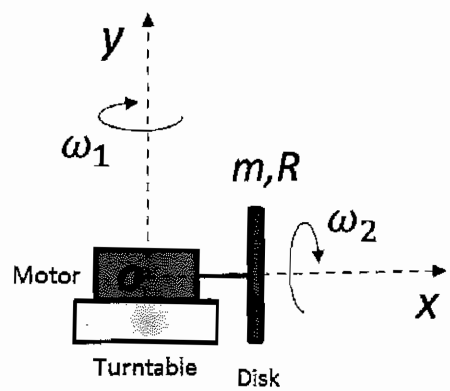

Figure 2

Consider point O as a fixed point of rotation, counterclockwise (CCW), upward and to the right directions as positive. The moment of inertia of the disk about its axis of rotation is $\frac{1}{2}mR^2$ and the shaft length is L.

a. Compute the reactions at O

b. Say the turntable starts to accelerate with a constant CCW angular acceleration α, compute the reactions at O

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DYNAMICS AND VIBRATIONS

Problem 3

In the vibration testing of a structure, an impact hammer with a load cell (force measuring transducer) is used to cause excitation, as in the fixture below. The load cell measures the impact force exerted by the hammer as it strikes the structure.

Assume that:

m = 5 kg **k = 2000 N/m** **c = 10 N-sec/m (damping in the walls)**

In many cases, providing only one impact to a structure using an impact hammer is difficult and a second impact takes place after the first (as shown in Figure 3). In this case, the applied force F(t) can be expressed as:

$$F(t) = F_1\,\delta(t) + F_2\,\delta(t-\tau)$$

where $\delta(t)$ is the Dirac delta function and τ is the time between impacts in seconds.

Assume that:

$F_1 = 20$ N **$F_2 = 10$ N** **$\tau = 0.2$ sec**

Find the complete response of the structure as a function of time.

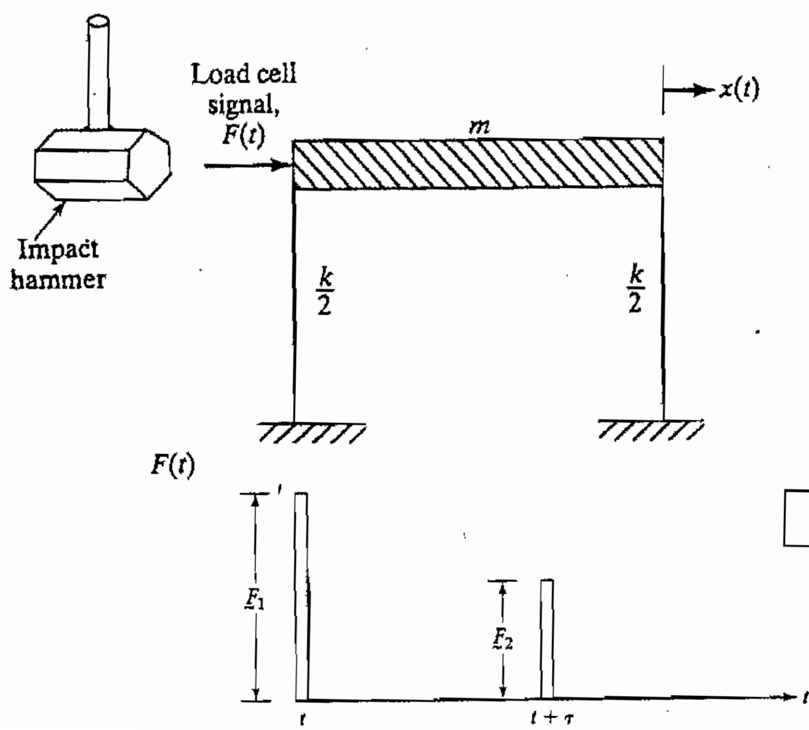

Figure 3

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DYNAMICS AND VIBRATIONS

Problem 1

Mass m_2 is pivoted at the center of mass m_1 by a rigid massless link of length R, see Figure 1. Neglect friction at the pivot. The motion is in the vertical plane. K is the spring stiffness, c is damping friction coefficient, and $F(t)$ is an external force acting on m_1. Choose y and θ as the generalized coordinates.

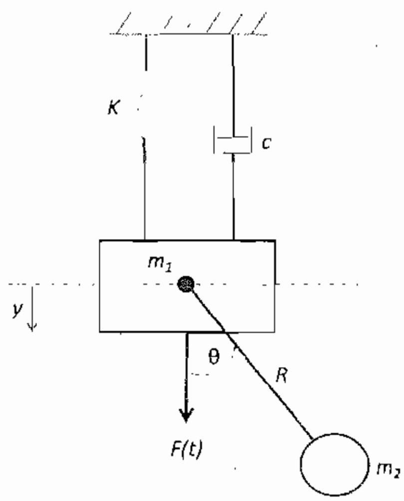

Figure 1

a. Obtain the Lagrange equations of motion in the absence of the damper and external force.

b. Now consider the damper (still $F(t)=0$), and assume a Rayleigh Dissipation Function $\mathcal{F} = \frac{1}{2}c\dot{\theta}^2$. Obtain the Lagrange equations of motion.

c. Now also consider the external force $F(t)$ and obtain the Lagrange equations of motion.

Note: No need to re-derive equations obtained in a previous section – use them directly.

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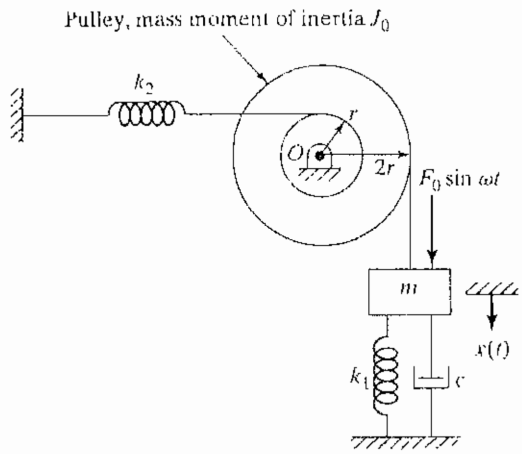
DYNAMICS AND VIBRATIONS

Problem 2

Find the equation of motion and the total response of the system shown in figure for the following data:

$k_1 = 1000$ N/m, $k_2 = 500$ N/m, $c = 500$ N.s/m, $m = 10$ kg,
$r = 5$ cm, $J_0 = 1$ kg.m^2, $F_0 = 50$ N, $\omega = 20$ rad/s.

I.C. $x_0 = 1$ m, $\dot{x}_0 = 0$ m/s^2



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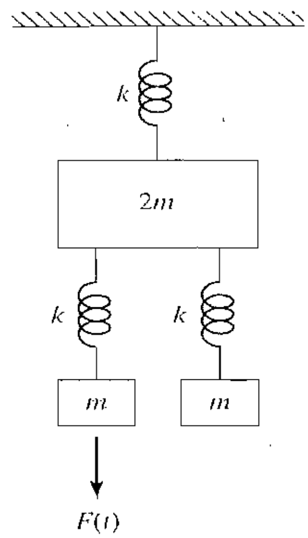
DYNAMICS AND VIBRATIONS

Problem 3

Determine the steady-state solutions of three masses in figure below when a harmonic force $F(t) = F_0 \sin(\omega t)$ is applied to the lower left mass with the following data:

$m = 1$ kg, $k = 1000$ N/m, $F_0 = 5$ N, $\omega = 10$ rad/s.



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Mechanical Engineering, Columbia University
DYNAMICS AND VIBRATIONS

Problem 1

A spring pendulum as shown in Figure 1 comprises a mass m suspended by an elastic spring of stiffness k and unstretched length r_0. Assume the motion is in the vertical plane, the suspension hinge is frictionless, and the generalized displacements are r and θ.

Figure 1

a. Derive the kinetic energy of the system

b. Derive the potential energy of the system

c. Derive the Lagrangian function L

d. Derive the equations of motion of the pendulum via Lagrange's equations

   If the hinge has friction:

e. What would then be the equations of motion?
   Hint: you may use the same L, as obtained above, in the Lagrange equations form that contain the Rayleigh Dissipation Function, F, and use $F = \frac{1}{2}c\dot{\theta}^2$, where c, is the coefficient of friction.

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Mechanical Engineering, Columbia University
DYNAMICS AND VIBRATIONS

Problem 2

The wheel and axle assembly of a locomotive can be represented as shown in Figure 1. k_a and b_a are the stiffness and damping coefficient of the axle, while k_w is the stiffness of the wheel. m_a and m_w are the axle and wheel masses respectively. The locomotive is moving horizontally, in the vertical plane where g is the acceleration of gravity.

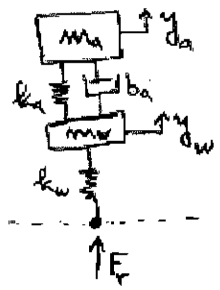

Figure 1

a. Draw a free-body diagram, or a linear graph, of this dynamic system, defining any new quantity that you may introduce.

b. Derive the vertical equations of motion of the axle and the wheel, assuming the rail provides impulsive forces, F_r, due to irregularities on its surface. Do not solve for the displacements.

c. To have the rail profile, $x(t)$, as the input, instead of forces as above, see Figure 2, we can consider the wheel and axle assembly as stationary and the rail as being pulled to the left from underneath it. How do you suggest solving for the displacements $y_a(t)$ and $y_w(t)$? Briefly outline the steps; do not solve for the displacements.

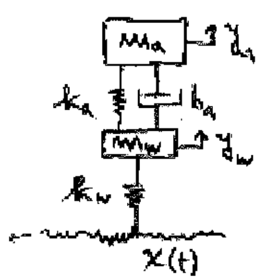

Figure 2

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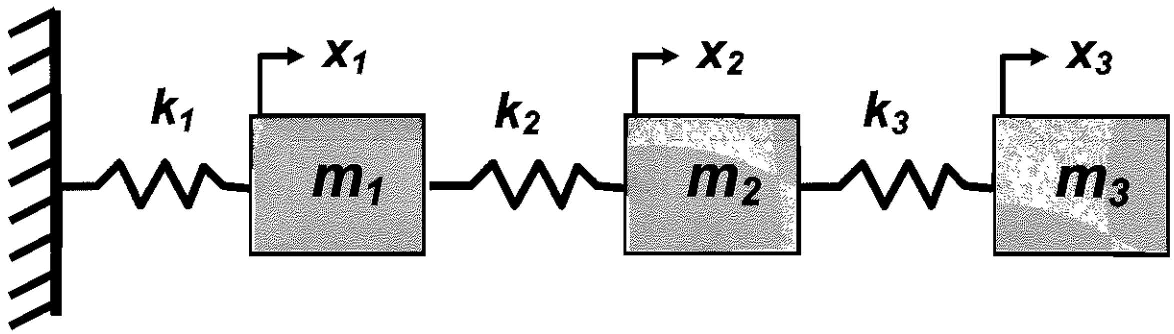
DYNAMICS AND VIBRATIONS

Problem 3

For the triple spring-mass system shown below, assume:

$$k_1 = k,\quad k_2 = 2\,k,\,,\quad k_3 = 3\,k,\quad m_1 = m,\quad m_2 = 2\,m,\,,\quad m_3 = 3\,m$$



a) Draw the free body diagram. Derive the linear equation of motion. State any assumptions that you make.

b) Find the natural frequencies and the mode shapes. Plot the mode shapes.

c) Find the complete response of the system as a function of time for the following initial conditions:

$$x_1(0) = x_2(0) = x_3(0) = 0,\quad v_1(0) = v_c\,,\quad v_2(0) = v_3(0) = 0$$

Note: $v_1(0)$ is the initial velocity of mass m_1 and v_c is a constant.

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Mechanical Engineering, Columbia University

DYNAMICS AND VIBRATIONS

Problem 1

One of the wheels and leaf springs of a automobile traveling over a rough road is shown in Figure 1 below. For simplicity, all of the wheels can be assumed to be identical and the system can be idealized as shown in Figure 2. The automobile has a mass of $\mathbf{m_1 = 1000\ kg}$ and the leaf springs have a total stiffness of $\mathbf{k_1 = 400\ k\ N/m}$. The wheels and axles have a mass of $\mathbf{m_2 = 300\ kg}$ and the tires have a stiffness of $\mathbf{k_2 = 500\ k\ N/m}$.

If the road surface varies with an amplitude $\mathbf{Y = 0.1}$ m and a period of $\boldsymbol{\ell} = \mathbf{6\ m}$, find the critical velocities of the automobile.

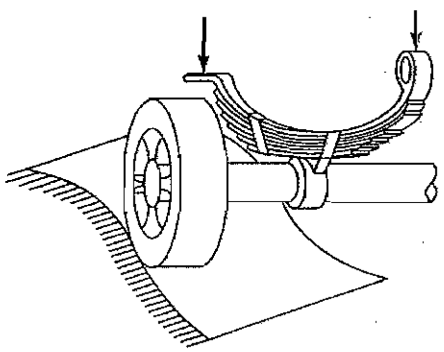

Figure 1

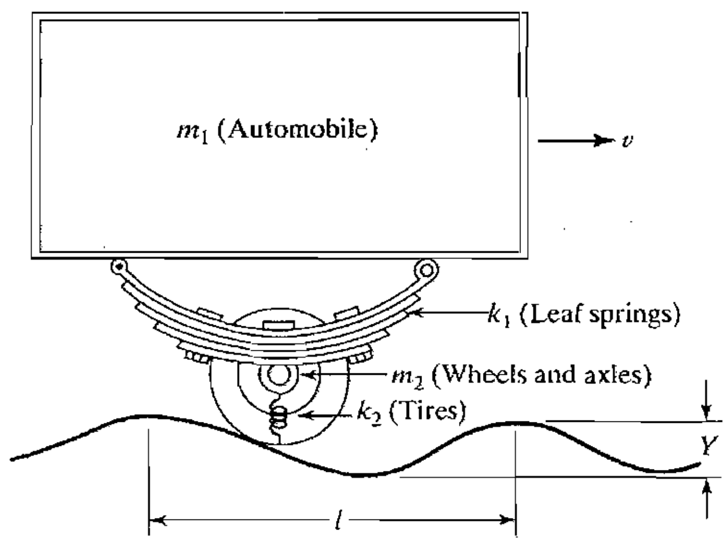

Figure 2

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DYNAMICS AND VIBRATIONS

Problem 2

For the system shown below, **x** denotes the absolute displacement of the cart.

a) Draw the free body diagram for the system labeling all the forces.

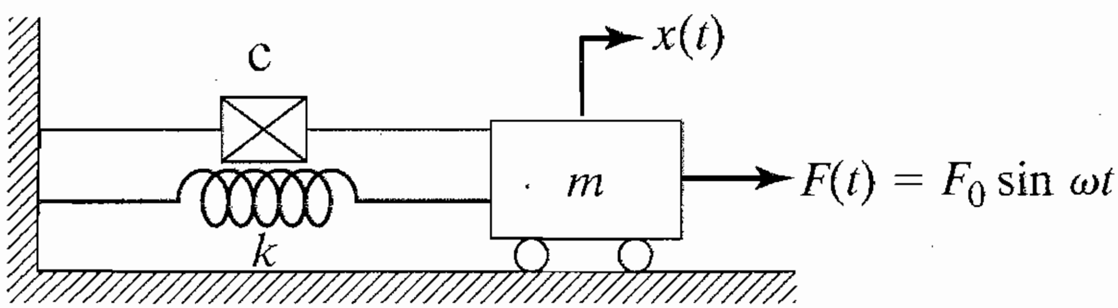
b) Using D'Alembert's principle, derive the equation of motion.

c) Determine the undamped natural frequency ω_n of the system and the damping coefficient ζ for the damping constant **c = 30 N-s/m** and the parameters below. State any assumptions that you make.

m = 100 kg **k = 2000 N/m** $\mathbf{F_0 = 20}$ **N** $\boldsymbol{\omega}$ **= 6.3 rad/sec**

d) For zero initial conditions and the values for the damping constant in part (c), determine the displacement **x** as a function of time.

e) Compute the value of c so that the steady-state response amplitude of the system is 0.01 m.



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DYNAMICS AND VIBRATIONS

Problem 3

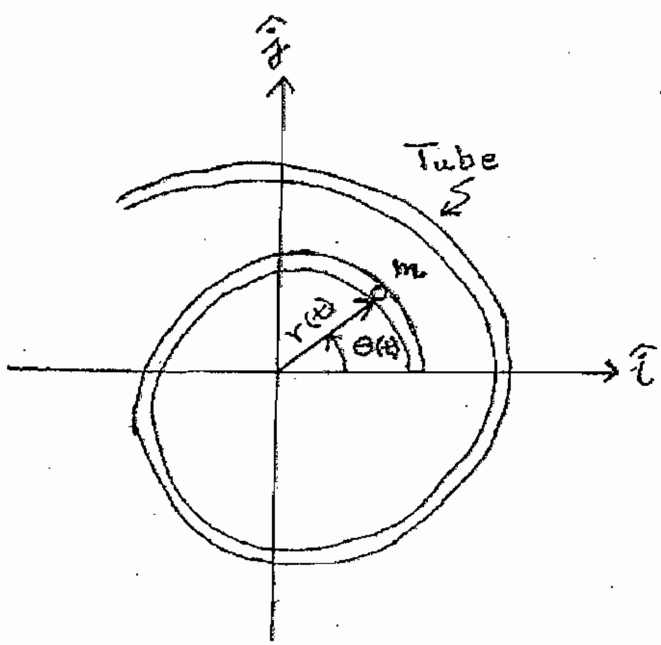
A mass of mass m travels inside a frictionless tube in the horizontal plane. Its position as a function of time is given in polar coordinates as

$$r(t) = 1 + t$$
$$\theta(t) = t$$
$$t \geq 0$$

as shown in the figure.

(a) Find the forces that must be applied to the mass so that it follows this path.

(b) What forces must the tube apply to the mass so that the mass has this motion in time?



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DYNAMICS AND VIBRATIONS

Problem 1

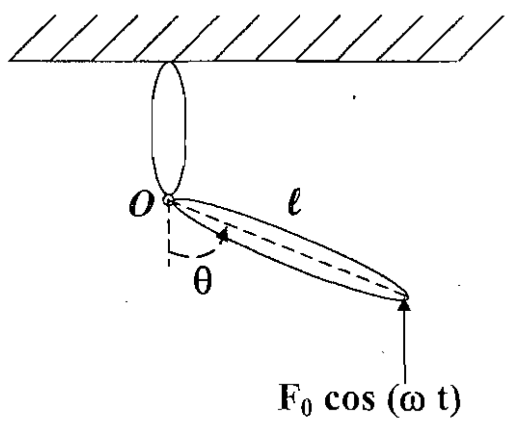
Vibration of body parts is a significant problem in designing machines. A jackhammer (shown below) provides a harmonic force to the operator's arm. To model this situation, treat the forearm as a compound pendulum subject to a harmonic excitation as illustrated below. Assume that the mass of the forearm is 5.0 kg and the length is 42.4 cm. Consider point *O* (the elbow) as a fixed pivot.

a) Draw the free body diagram for the forearm labeling all forces and torques.

b) Derive the linear equation of motion. State any assumptions that you make. (Hint: recall that the moment of inertia $\mathbf{J_0}$ of a slender rod pivoted about its end is $\frac{1}{3}\mathbf{m}\,\ell^2$ where ℓ is the length of the rod.)

c) Solve for the steady-state response for the hand end of the arm as a function of time if the jackhammer applies a harmonic force on each forearm of 10 N at 2 Hz (i.e. $\mathbf{F_0}$ = 10 N). Compute the maximum deflection of the hand.

d) Assume the force applied by the jackhammer is reduced to 2 N and the frequency changes so that it is less than the natural frequency of the forearm and the hand exhibits a beat frequency of 0.2 Hz. What is the maximum deflection of the hand? Are the assumptions used in part b still valid?



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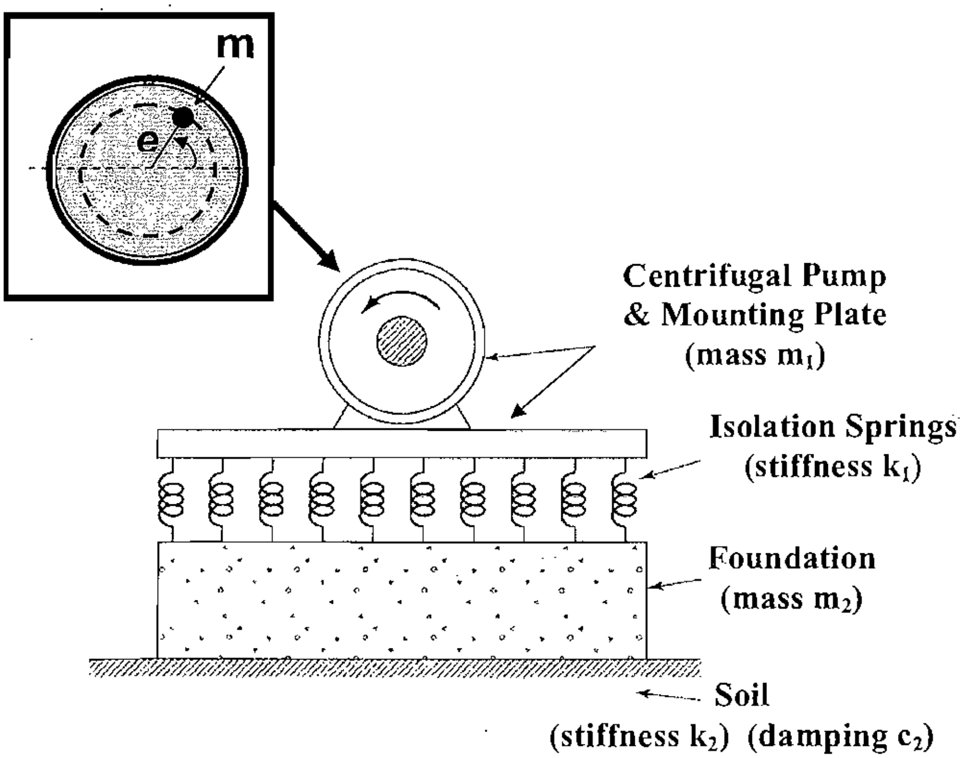
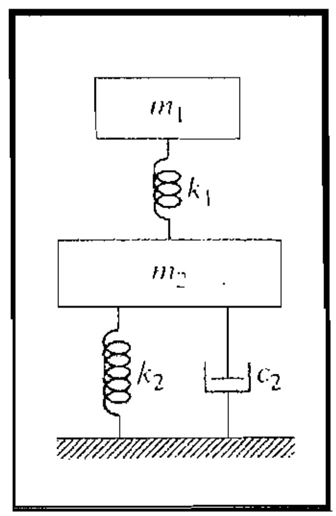
DYNAMICS AND VIBRATIONS

Problem 2

A centrifugal pump mounted rigidly to a plate has an unbalanced mass **m** which rotates through an eccentricity **e**. The pump is supported on a rigid foundation of mass $\mathbf{m_2}$ through a set of isolator springs of total stiffness $\mathbf{k_1}$ as shown below. The isolator springs constrain the pump to only move in the vertical direction. The mass of the pump and mounting plate is $\mathbf{m_1}$. Assume that when the pump starts (time zero), the unbalanced mass is at the extreme lowest point of its travel and both the foundation and the plate are at equilibrium. If the soil on which the foundation sits has a stiffness $\mathbf{k_2}$ and damping $\mathbf{c_2}$, find the displacement of the foundation and the pump as functions of time for the following data:

m = 0.25 kg	$\mathbf{m_1}$ **= 360 kg**	$\mathbf{m_2}$ **= 900 kg**
e = 2.5 cm	**pump speed = 1200 RPM**	
$\mathbf{k_1}$ **= 350,000.0 N/m**	$\mathbf{k_2}$ **= 175,000.0 N/m**	$\mathbf{c_2}$ **= 35,000.0 N-s/m**





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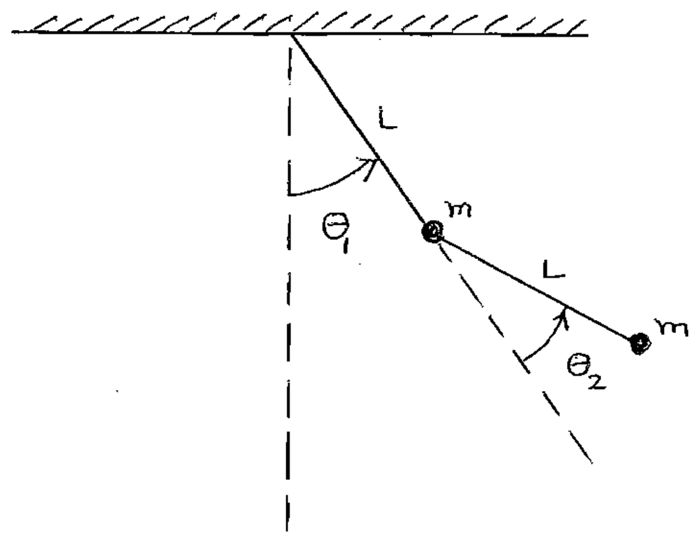
DYNAMICS AND VIBRATIONS

Problem 3

Consider the double pendulum under gravity as shown below. The angle of the upper pendulum is with respect to the local vertical, and the angle of the lower pendulum is with respect to the angle of the upper pendulum as shown. Each pendulum is considered an ideal pendulum with all its mass at the end, each mass denoted by m. The lengths of the two links are the same and this length is denoted by L. Use Lagrange's equation to develop the equations of motion of the two links.



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DYNAMICS AND VIBRATIONS

Problem 1

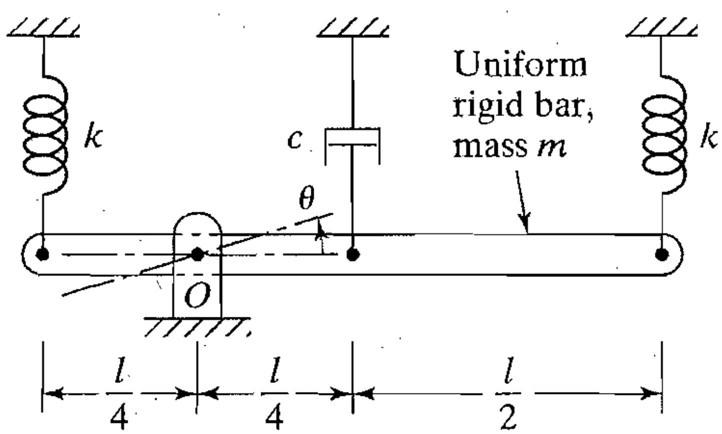
For the system shown in part (a) below:

a) Draw the free body diagram for the system labeling all the forces.

b) Derive the linear equation of motion. State any assumptions that you make.

c) Determine the undamped natural frequency ω_n (in rad/sec) of the system and the (dimensionless) damping coefficient ζ for the following parameters. (Hint: recall that the moment of inertia J_0 of a slender rod pivoted about its center is $\frac{1}{12} m l^2$ where l is the length of the rod.)

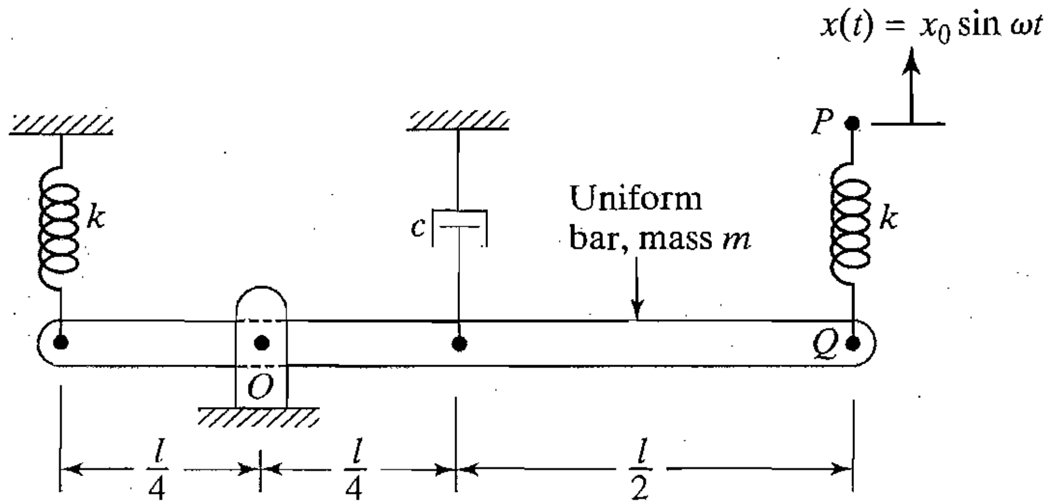
m = 10 kg **k = 1000 N/m** **c = 500 N-s/m** **l = 1 m**



(part a)

d) Replace the fixed mount on the right with a harmonic displacement at point P as shown in part (b) below. Using the method of undetermined coefficients compute the magnitude and phase of the steady-state response of the system (i.e. the particular solution of the equation of motion) for

$$\mathbf{x_0 = 1\ cm} \qquad \boldsymbol{\omega = 10\ rad/sec}$$



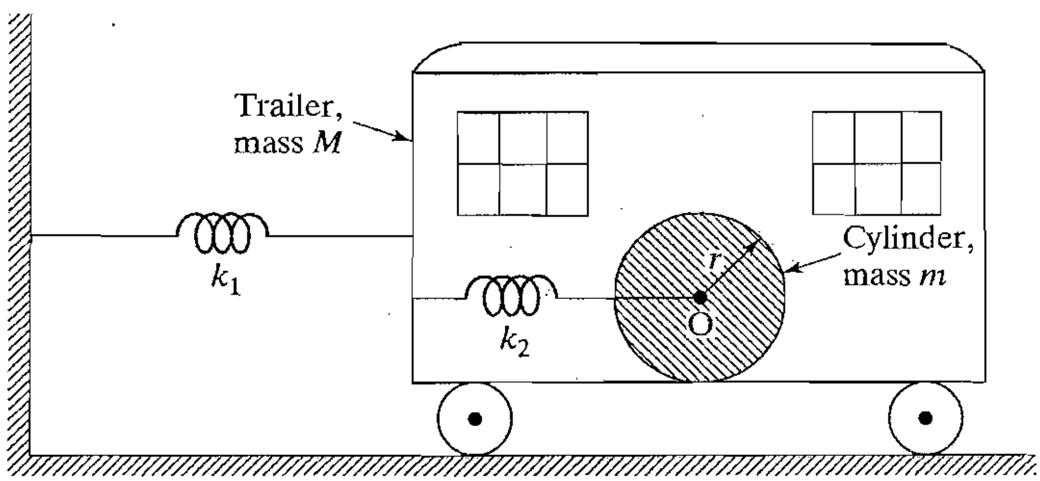
(part b)

Doctoral Qualifying Examination, January 2010
Mechanical Engineering, Columbia University

DYNAMICS AND VIBRATIONS

Problem 2

A trailer of mass **M** is connected to a rigid wall with a spring of stiffness $\mathbf{k_1}$ and can move on a frictionless horizontal surface, as shown below. A uniform cylinder of mass **m** connected to the wall of the trailer with a spring of stiffness $\mathbf{k_2}$ can roll on the floor of the trailer without slipping. Derive the equations of motion of the system. State any assumptions that you make.



DYNAMICS AND VIBRATIONS

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Department of Mechanical Engineering

Problem 3

A vector $\underline{r}$ can be written showing explicitly the coordinate system used for the components, so the vector $\underline{r}$ could be written in terms of components on the orthogonal unit vectors $\hat{i}_1, \hat{j}_1, \hat{k}_1$ as

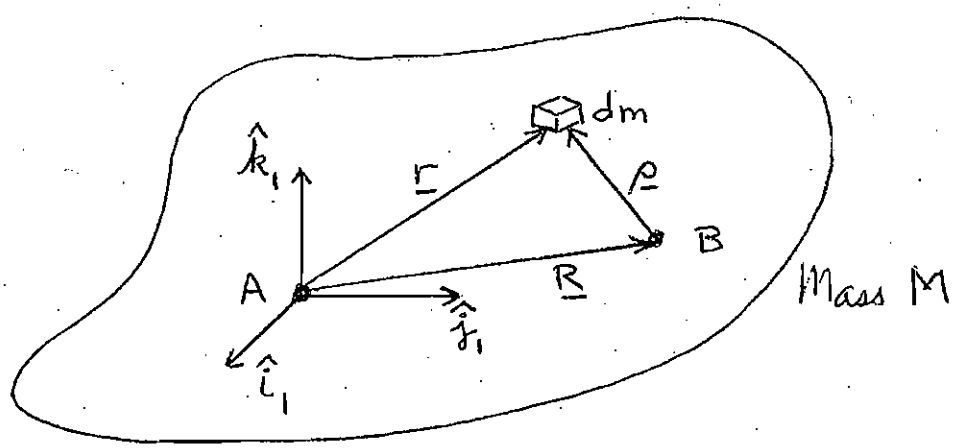
$$\underline{r} = r_1\hat{i}_1 + r_2\hat{j}_1 + r_3\hat{k}_1 = \begin{bmatrix} \hat{i}_1 & \hat{j}_1 & \hat{k}_1 \end{bmatrix}\begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} \qquad (1)$$

We can also just write the components as entries in the column matrix

$$r = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} \qquad (2)$$

where we do not put an underbar under r when we refer to the matrix. In this form we have to remember what coordinate system we are using.

Consider the mass M with differential element of mass dm in the following diagram.



The inertia matrix of this mass about point A written with respect to the reference frame $\hat{i}_1, \hat{j}_1, \hat{k}_1$ is given by

DYNAMICS AND VIBRATIONS

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Problem 3 (continued)

$$I_A = \int_M \left(\begin{bmatrix} r_1 & r_2 & r_3 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} I - \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} \begin{bmatrix} r_1 & r_2 & r_3 \end{bmatrix} \right) dm$$

where I is the identity matrix.

(i) Prove that this matrix is symmetric.

(ii) Develop an expression that gives the inertia matrix about point B written relative to these same coordinates $\hat{i}_1, \hat{j}_1, \hat{k}_1$.

(iii) Usually the inertia matrix is written as a 3×3 matrix of scalars, and this form corresponds the vector form (2) above where you have to remember what coordinate system it is written in. The equivalent of equation (1) which explicitly states the coordinates is given by

$$\underline{\underline{I}}_A = \begin{bmatrix} \hat{i}_i & \hat{j}_1 & \hat{k}_1 \end{bmatrix} I_A \begin{bmatrix} \hat{i}_1 \\ \hat{j}_1 \\ \hat{k}_1 \end{bmatrix}$$

This time two underbars are used to indicate this version compared to the simple matrix I_A.

Suppose that $I_A = \begin{bmatrix} 50 & 0 & 0 \\ 0 & 40 & -10 \\ 0 & -10 & 30 \end{bmatrix}$. Find a matrix A representing a change of coordinates

$$\begin{bmatrix} \hat{i}_2 \\ \hat{j}_2 \\ \hat{k}_2 \end{bmatrix} = A \begin{bmatrix} \hat{i}_1 \\ \hat{j}_1 \\ \hat{k}_1 \end{bmatrix}$$

such that the resulting new I_A for the new coordinates is diagonal. Explain your work, and draw a picture showing the relationship between the initial coordinates and the new coordinates.

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Mechanical Engineering, Columbia University

DYNAMICS AND VIBRATIONS

Problem 1:

For the system shown below, **x** and **y** denote, respectively, the absolute displacement of the point **Q** and the mass **m**.

a) Draw the free body diagram for the system labeling all the forces.

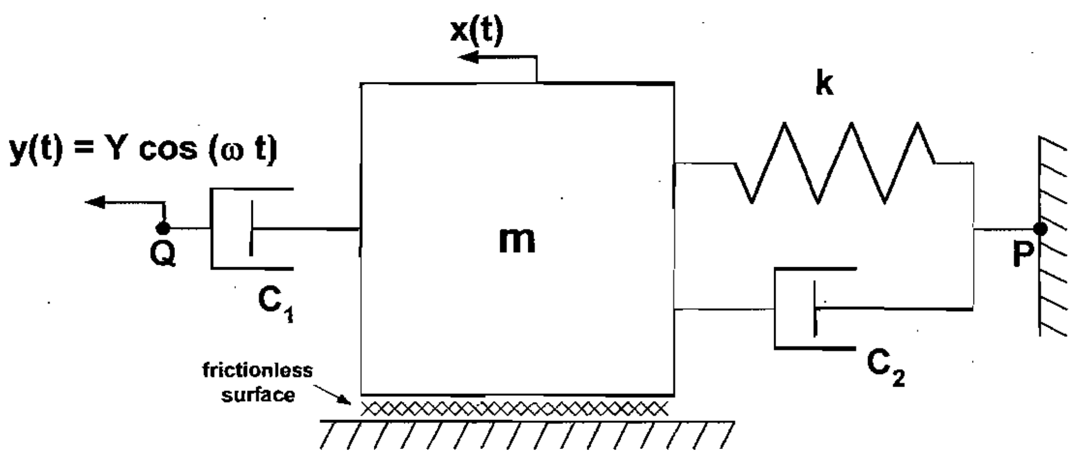
b) Using D'Alembert's principle, derive the equation of motion.

c) Determine the undamped natural frequency ω_n of the system and the damping coefficient ζ for the following parameters. State any assumptions that you make.

$\mathbf{m = 10\ kg}$	$\mathbf{k = 4000\ N/m}$	$\mathbf{c_1 = 15\ N\text{-}s/m}$	$\mathbf{c_2 = 5\ N\text{-}s/m}$
$\mathbf{Y = 0.1\ m}$	$\mathbf{\omega = 10\ rad/sec}$		

d) For zero initial conditions and the values for the parameters in part (c), determine the displacement **x** as a function of time.

e) For zero initial conditions and the values for the parameters in part (c), determine the force transmitted to the wall at the point **P** as a function of time.



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DYNAMICS AND VIBRATIONS

Problem 2:

In the torsional system shown below, two disks are mounted on a shaft. The three shaft segments have spring stiffness $\mathbf{k_{T1}}$, $\mathbf{k_{T2}}$ and $\mathbf{k_{T3}}$ as indicated. The disks have mass moment of inertia $\mathbf{J_1}$ and $\mathbf{J_2}$.

a) Derive the equations of motion using energy methods for the coordinates θ_1 and θ_2 (show all work).

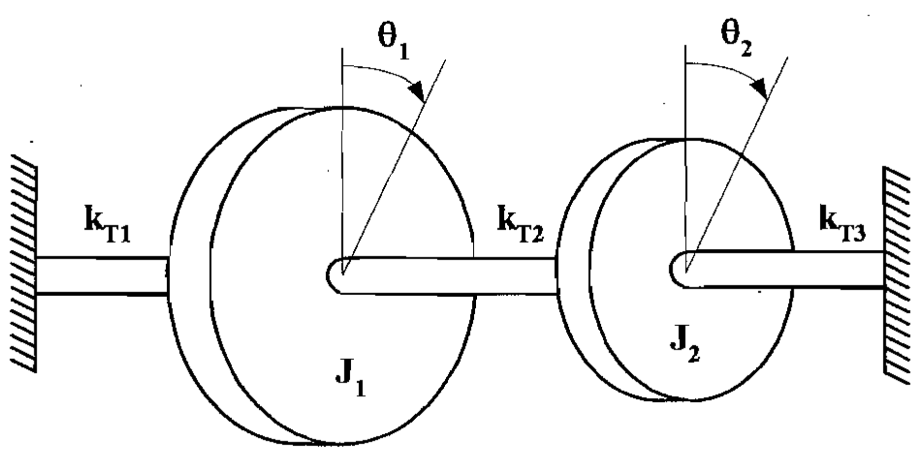
b) Find the natural frequencies and mode shapes for the system if $\mathbf{J_1 = 2\,J_0}$, $\mathbf{J_2 = J_0}$ and $\mathbf{k_{T1} = k_{T2} = k_{T3} = k_T}$

c) Sketch the mode shapes.

Hint: Lagrange's equation is:

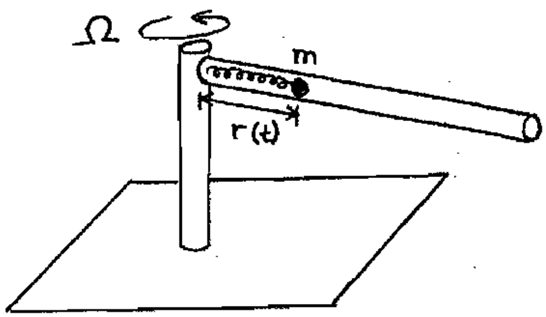
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) + \frac{\partial D}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = Q_i$$

where L = the Lagrangian = K – P
K = kinetic energy
P = potential energy
D = power dissipation
q_i = a generalized coordinate
Q_i = the externally applied forces or moments



DYNAMICS AND VIBRATIONS

Problem 3:



Mass *m* at the end of an ideal spring of length L and spring constant K, can move radially inside a frictionless tube that rotates in the horizontal plane with constant angular velocity Ω.

a) Consider all possible positive spring constants and angular velocities, and find the equilibrium location for the mass, i.e. the radial distance r from the axis of rotation at which the mass could stay for all time.

b) Write the differential equation for the radial motion of the mass as a function of time. Explain your reasoning.

c) Suppose that at time zero the mass is at radial distance along the tube from the axis of rotation, $r(0) = L$, with no radial velocity. Find the radial distance to the mass as a function of time $r(t)$. Give this result in terms of real valued functions, and give the result for all possible positive spring constants and angular velocities.

d) In the solution of part c of this question, under what conditions does the mass oscillate and when does it not oscillate? When it oscillates, what is the amplitude of the oscillation, and how does it relate to the unstretched length of the spring and the equilibrium location? When it does not oscillate what types of motion are possible?

e) Now consider that instead of the tube spinning in the horizontal plane, it is spinning in the vertical plane so that gravity influences the radial motion of the mass. Give the solution as in part c for the case when the initial condition at time zero applies when the tube happens to be hanging straight down.

DYNAMICS AND VIBRATIONS

Doctoral Qualifying Examination, January 2008
Mechanical Engineering, Columbia University

Problem 1

A single cylinder air compressor weighing 1000 N is mounted on rubber mounts as shown below. The stiffness and damping constants of the rubber mounts are measured to be 10^6 N / m and 2000 N – sec / m, respectively. The unbalance of the compressor is equivalent to a mass of 0.1 kg located at the end of the crank (i.e. at point A).

Consider the air compressor to be a single degree-of-freedom system in the vertical direction.

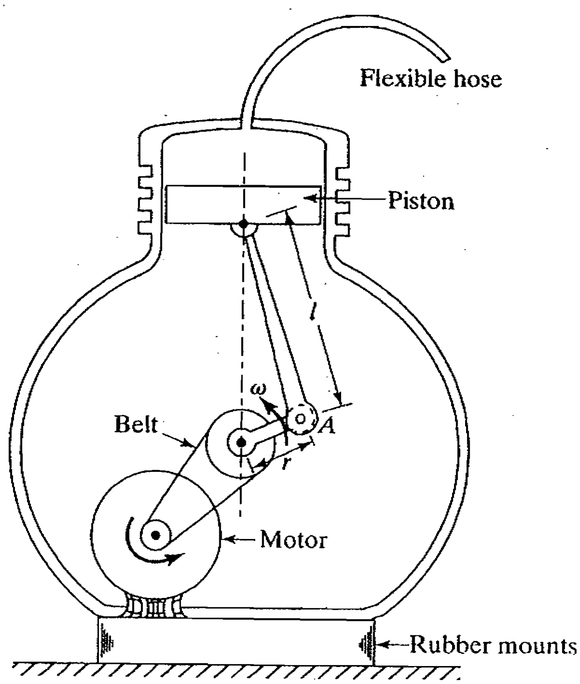
a) Draw the free body diagram for the system labeling all the forces.

b) Using D'Alembert's principle, derive the equation of motion.

c) Determine the undamped natural frequency ω_n and the damping coefficient ζ.

d) Assume that the crank is initially horizontal with r = 10 cm and l = 40 cm. Determine the complete response of the air compressor to an initial displacement of 1 mm, zero initial velocity and a crank speed of 3000 rpm.

e) At what crank speed (in rpm) will the air compressor exhibit the maximum steady-state displacement? What is the amplitude of this steady-state displacement?



Common Laplace Transform Pairs

Time Domain Function		Laplace Domain Function
Name	**Definition***	
Unit Impulse	$\delta(t)$	1
Unit Step	$u(t)$	$\frac{1}{s}$
Unit Ramp	t	$\frac{1}{s^2}$
Exponential	e^{-at}	$\frac{1}{s+a}$
Asymptotic Exponential	$\frac{1}{a}(1-e^{-at})$	$\frac{1}{s(s+a)}$
Dual Exponential	$\frac{1}{b-a}\left(e^{-at}-e^{-bt}\right)$	$\frac{1}{(s+a)(s+b)}$
Asymptotic Dual Exponential	$\frac{1}{ab}\left[1+\frac{1}{a-b}\left(be^{-at}-ae^{-bt}\right)\right]$	$\frac{1}{s(s+a)(s+b)}$
Time multiplied Exponential	te^{-at}	$\frac{1}{(s+a)^2}$
Sine	$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
Cosine	$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
Decaying Sine	$e^{-at}\sin(\omega t)$	$\frac{\omega}{(s+a)^2+\omega^2}$
Decaying Cosine	$e^{-at}\cos(\omega t)$	$\frac{s+a}{(s+a)^2+\omega^2}$
Generic Oscillatory Decay	$e^{-at}\left[B\cos(\omega_n t)+\frac{C-aB}{\omega_n}\sin(\omega_n t)\right]$	$\frac{Bs+C}{(s+a)^2+\omega_n^2}$
Prototype Second Order Lowpass, underdamped	$\frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1-\zeta^2}\,t\right)$	$\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$
Prototype Second Order Lowpass, underdamped - Step Response	$1-\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1-\zeta^2}\,t+\phi\right)$ $\phi=\tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$	$\frac{\omega_n^2}{s(s^2+2\zeta\omega_n s+\omega_n^2)}$

DYNAMICS AND VIBRATIONS

Doctoral Qualifying Examination, January 2008
Mechanical Engineering, Columbia University

Problem 2

Two identical pendulums, each with a mass m and length l, are connected by a spring of stiffness k at a distance d from the fixed end as show below.

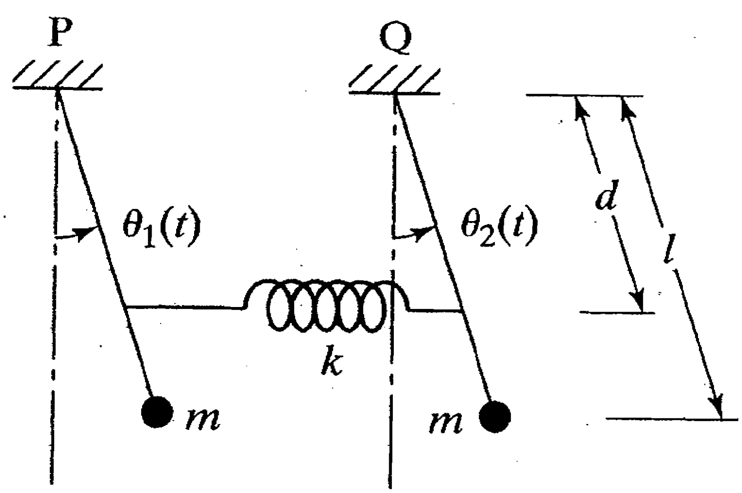
a) Derive the equations of motion for the two masses. (Hint: assume small angles.)

b) Find the natural frequencies and mode shapes for the system.

c) Find the free vibration response of the system for the initial conditions

$$\theta_1(0) = a \qquad \theta_2(0) = 0 \qquad \text{zero initial angular velocities}$$

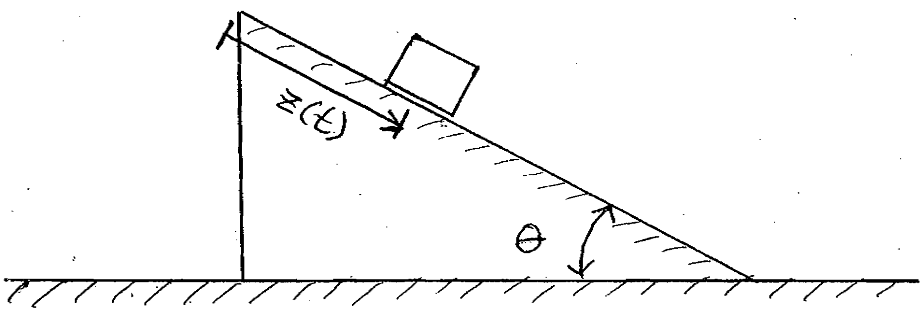
d) Determine the condition(s) under which the system exhibits a beating phenomenon.



DYNAMICS AND VIBRATIONS **Doctoral Qualifying Examination, January 2008**
Mechanical Engineering, Columbia University

Problem 3

Consider a block of mass m at position $z = 0$ at time $t = 0$, on a plane inclined at an angle θ, as in the accompanying figure ($\dot{z}(0) = 0$)



There are various different models for friction:

(i) Viscous friction where the friction force is proportional to the velocity
(ii) Simple Coulomb friction where the friction force is a maximum of the coefficient of friction μ times the normal force
(iii) A stiction model in which the friction force can reach μ_S times the normal force when there is no relative motion, but reduces to μ times the normal force when there is relative motion, $\mu_S > \mu$.

Model the block as a point mass, and consider each of these three models. Give the position of the block as a function of time for $t > 0$, and do this for all possible angles $0 < \theta < 90$ degrees.

DYNAMICS AND VIBRATIONS **Doctoral Qualifying Examination, January 2007**
Mechanical Engineering, Columbia University

Problem 1
A propeller system for a ship is shown below. The system consists of the propeller, drive train, engine and flywheel. The diameter ratio of the gears is $D_1 / D_2 = 1.5$ (i.e. gear 2 makes 3 rotations to every 2 rotations of gear 1). The moment of inertias of gear 1 and gear 2 are 500 and 100 kg-m^2, respectively. The flywheel, engine and propeller moment of inertias are 10^4, 10^3 and 2500 kg-m^2, respectively. The torsional stiffness of shaft 1 and shaft 2 are $5\text{x}10^6$ and 10^6 N-m/rad, respectively.

Since the shaft between the engine and gears is short, assume that it is very stiff compared to the other shafts. Also, neglect shaft inertias and all damping.

Derive the equations of motion using energy methods indicating the generalized coordinates (show all work). Obtain the natural frequencies and mode shapes of the system. Sketch the mode shapes.

Hint: Lagrange's equation is:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) + \frac{\partial D}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = Q_i$$

where L = the Lagrangian = K – P

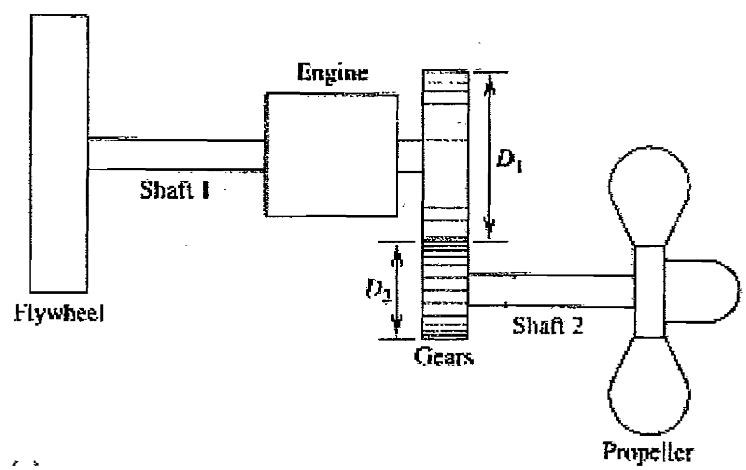
K = kinetic energy

P = potential energy

D = power dissipation

q_i = a generalized coordinate

Q_i = the externally applied forces or moments



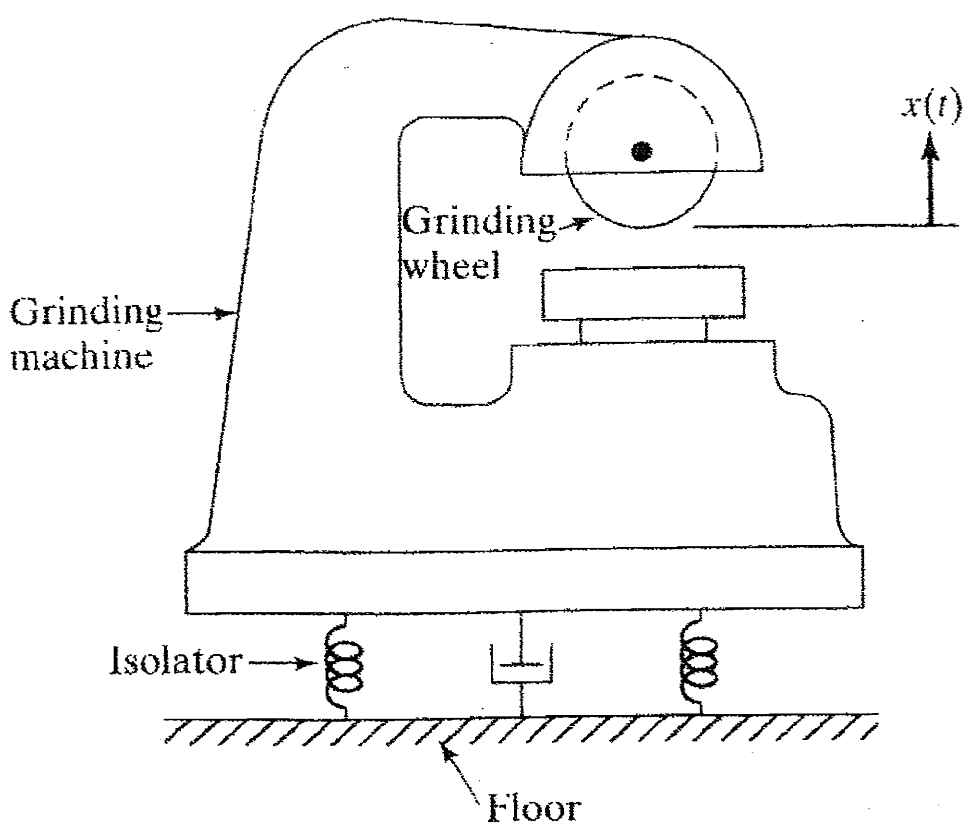
DYNAMICS AND VIBRATIONS **Doctoral Qualifying Examination, January 2007**
Mechanical Engineering, Columbia University

Problem 2
The precision grinding machine shown is supported on an isolator that has a total stiffness of $1x10^6$ N/m and a total viscous damping constant of 1000 N-s/m. Assume that the grinding machine and the wheel are a rigid body of weight 5000 N.

a) The floor on which the machine is mounted is subjected to a harmonic disturbance at a frequency of 200 Hz due to the operation of an unbalanced engine in the vicinity of the grinding machine. Find the maximum acceptable displacement amplitude of the floor if the resulting amplitude of vibration of the grinding wheel is to be restricted to 10^{-6} m.

b) With the unbalanced engine turned off and the grinding machine at equilibrium, a 20 kg piece of steel plate is placed on the grinding machine bed. Model the loading of the steel plate on the bed as a step at time zero and predict the response of the grinding machine in one degree of freedom (i.e. x(t) in the figure).

(Hint: use table of Laplace Transform pairs.)



Common Laplace Transform Pairs

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Prototype Second Order Lowpass, underdamped	$\frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1-\zeta^2}\,t\right)$	$\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$
Prototype Second Order Lowpass, underdamped - Step Response	$1-\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1-\zeta^2}\,t+\phi\right)$ $\phi=\tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$	$\frac{\omega_n^2}{s(s^2+2\zeta\omega_n s+\omega_n^2)}$

*All time domain functions are implicitly=0 for t<0 (i.e. they are multiplied by unit step).

DYNAMICS AND VIBRATIONS

Doctoral Qualifying Examination, January 2007
Mechanical Engineering, Columbia University

Problem 3

Let $\hat{i}$, $\hat{j}$, $\hat{k}$ be the unit vectors along the axes of an inertial coordinates system on the earth with $\hat{k}$ being vertical. At time $t = 0$ a point mass projectile of mass m has position vector $\underline{x}$ and velocity vector $\underline{v}$ given by

$$\underline{x}(0) = x_1(0)\hat{i} + x_1(0)\hat{j} + x_1(0)\hat{k}$$
$$\underline{v}(0) = v_1(0)\hat{i} + v_1(0)\hat{j} + v_1(0)\hat{k}$$

Suppose temporarily that there is no air resistance for this point mass.
(A) What are the differential equations of motion of this the point mass.
(B) Find the velocity components of the point mass motion as a function of time.
(C) Find the position components of the point mass motion as a function of time.
(D) If the mass does not collide with the earth, what are the velocity components and the position components in the limit as time goes to infinity? Explain your result in physical terms.

Now consider that there is air resistance that is linear with velocity with coefficient b.
(E) Repeat parts (A), (B), (C), (D) for this case.

Now consider that the initial position is at the origin, and that the initial velocity is also zero. And suppose that the air resistance is proportional to the speed squared with proportionality constant b.
(F) Find a relation between the velocity of the point mass and time.
(G) What can you say about the velocity of the point mass in the limit as time goes to infinity?

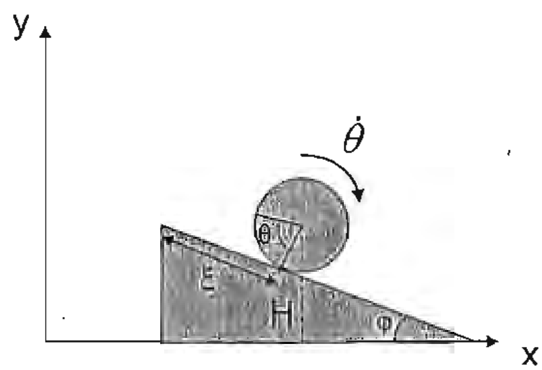
Note: $\tanh(u) = \dfrac{e^u - e^{-u}}{e^u + e^{-u}}$; $\coth(u) = \dfrac{1}{\tanh(u)}$; $\displaystyle\int \frac{dx}{x^2 - a^2} = -\frac{1}{a}\coth^{-1}\left(\frac{x}{a}\right)$

DYNAMICS AND VIBRATIONS

Doctoral Qualifying Examination, January 2006
Mechanical Engineering, Columbia University

Problem 1:

A sphere of mass M and radius R rolls without slipping down a triangular block of mass m that is free to move on a frictionless horizontal surface, as shown below.



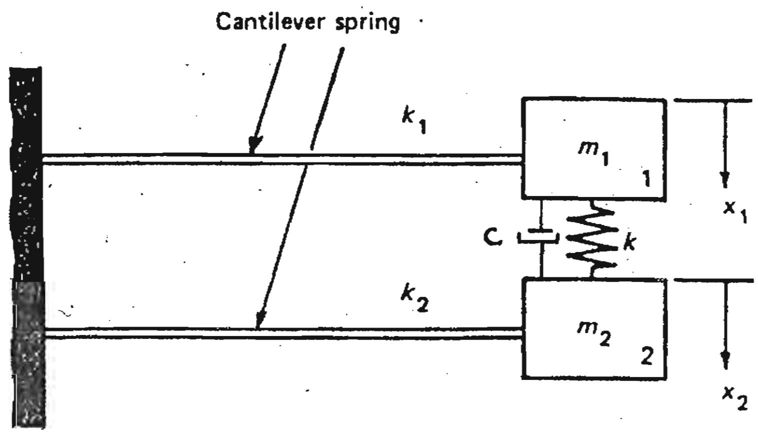
(a) Find the Lagrangian and state Lagrange's equations for this system subject to the force of gravity at the surface of the earth.
(b) Find the motion of the system by integrating Lagrange's equation, given that all objects are initially at rest and the sphere's center is initially at a distance H above the surface.

DYNAMICS AND VIBRATIONS

Doctoral Qualifying Examination, January 2006
Mechanical Engineering, Columbia University

Problem 2:

Consider a two degree of freedom system shown below



a) Derive the governing equations of motion (treat the cantilever beams as equivalent springs k_1 and k_2).

b) Obtain natural frequencies and modes for the case k_1=k_2=k, m_1=m_2=m.

DYNAMICS AND VIBRATIONS

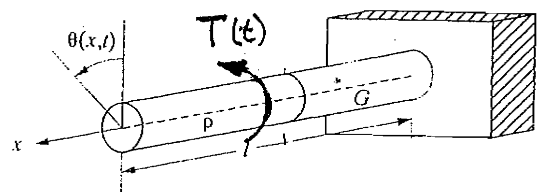
Doctoral Qualifying Examination, January 2006
Mechanical Engineering, Columbia University

Problem 3:

Consider torsional vibrations of a uniform shaft of length ℓ that is fixed at x = 0 and free at x = ℓ. A steady-state torque is acting at x = $\ell/2$.
Let the torque be represented as

$$T(t) = A \sin \omega t$$

where A is torque amplitude and ω is the radial frequency. Assume the initial conditions are zero and neglect damping effects.



Given the partial differential equation of torsional vibrations,

$$GJ\frac{\partial^2\theta}{\partial x^2}(x,t) + f(x,t) = I_o\frac{\partial^2\theta}{\partial t^2}(x,t)$$

where GJ - torsional stiffness, G – sheer modulus, J – polar moment of area of the cross section, I_0 – mass polar moment of inertia, $I_0 = \rho J$, ρ - material mass density and

$$f(x,t) = T(t)\,\delta(x-\ell/2)$$

where δ denotes a delta function.

a) Obtain expressions for natural frequencies and mode shapes.
b) Using modal analysis and the orthogonality conditions of the modes obtained in part (a), determine steady-state response θ (x,t).

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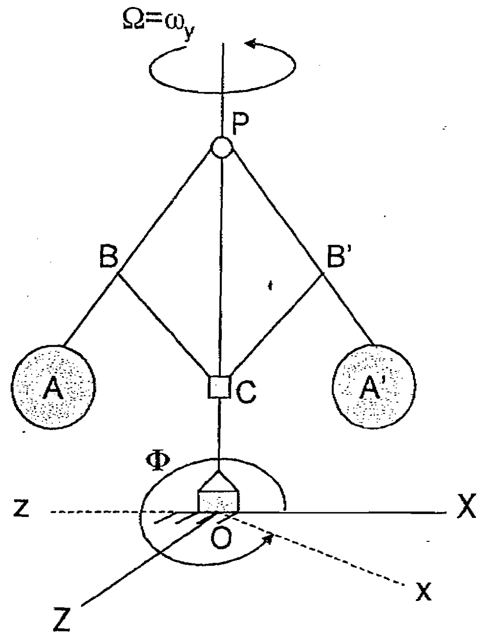
DYNAMICS AND VIBRATIONS

Doctoral Qualifying Examination, January 2005
Mechanical Engineering, Columbia University

Problem 1

A flyball governor consists of two heavy balls, A and A', four rigid bars of negligible mass and inertia, AP, A'P, BC and B'C, hinged at P and B, and a collar, C, which is free to slide on a vertical shaft OP that rotates with angular velocity Ω.



1. Derive the equations of motion for the flyball governor for the case when the angular velocity is constant.

2. Generalize the equations to handle the case when the angular velocity is a function of time $\Omega(t)$.

3. Obtain an expression for the heights of the balls as a function of speed in steady state operation. Would the solution to your equations developed in part 1 converge to this steady operation solution when integrated numerically? If not, say how you might modify the equations to do so.

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DYNAMICS AND VIBRATIONS

Doctoral Qualifying Examination, January 2005
Mechanical Engineering, Columbia University

Problem 2

Consider a projectile of mass one (MKS units) launched at time zero from horizontal position x=0 and vertical position y=0, with an initial horizontal velocity component of one and an initial vertical velocity component of one.

(i) First consider the case where there is no air resistance. Give the velocity and displacement of the projectile as a function of time for all positive time (assuming no collision with the earth).

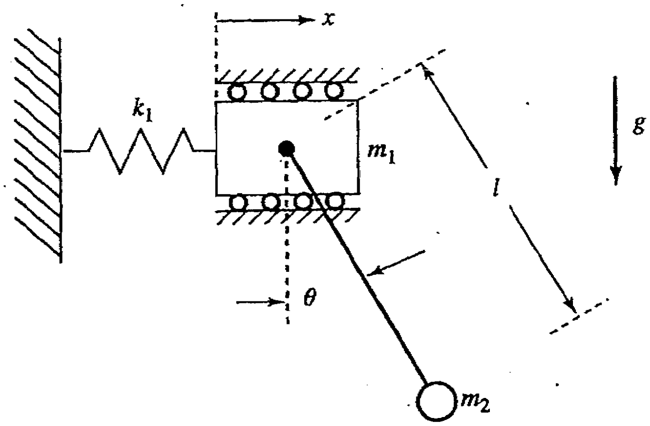
(ii) Now consider that there is air resistance that is proportional to the velocity, with a proportionality constant of unity. Give the velocity and the position as a function of time for all positive time. What happens to the x position as time gets large? If the projectile is launched off the edge of an infinitely tall cliff, what does the vertical velocity history look like? Does it keep accelerating forever under gravity? Can you obtain this answer for large time behavior by some direct logic?

(iii) A better model of air resistance is as a force that is proportional to the square of velocity. Again assume the proportionality constant is unity. Again give the horizontal and vertical position and velocity as a function of time, and analyze their behavior for large time.

Problem 3

(a) Using Lagrange method to derive the equations of motion for the system shown below. Note that gravity cannot be neglected and θ can be assumed small.

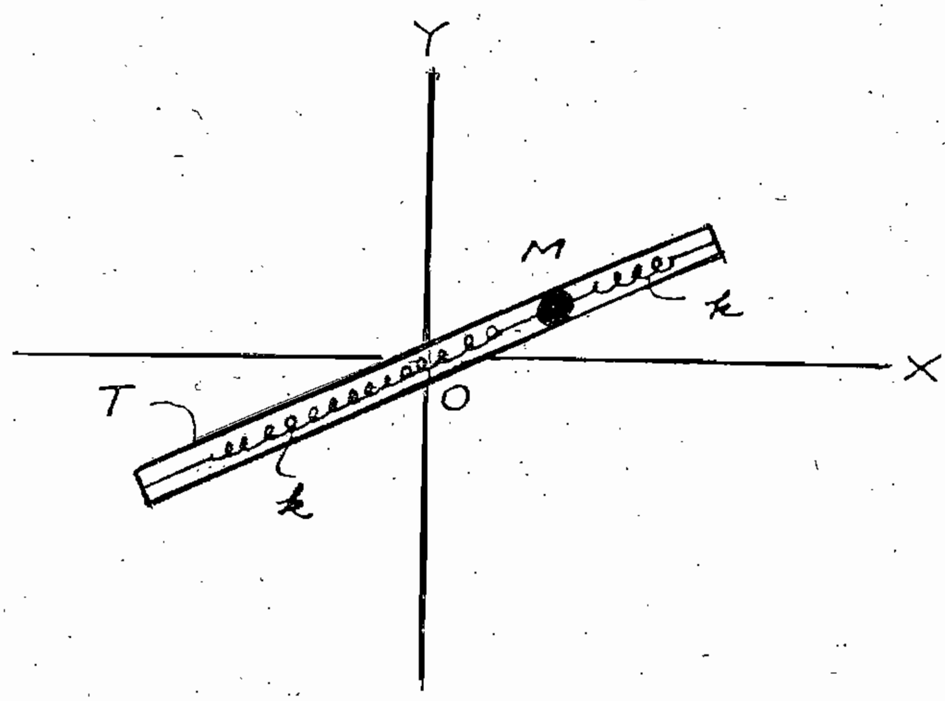
(b) Determine the natural frequencies and mode shapes of vibration of the system. $m_1 = 10$ kg, $m_2 = 1$ kg, $l = 1$ m, and $k_1 = 15$ N/m.



DYNAMICS

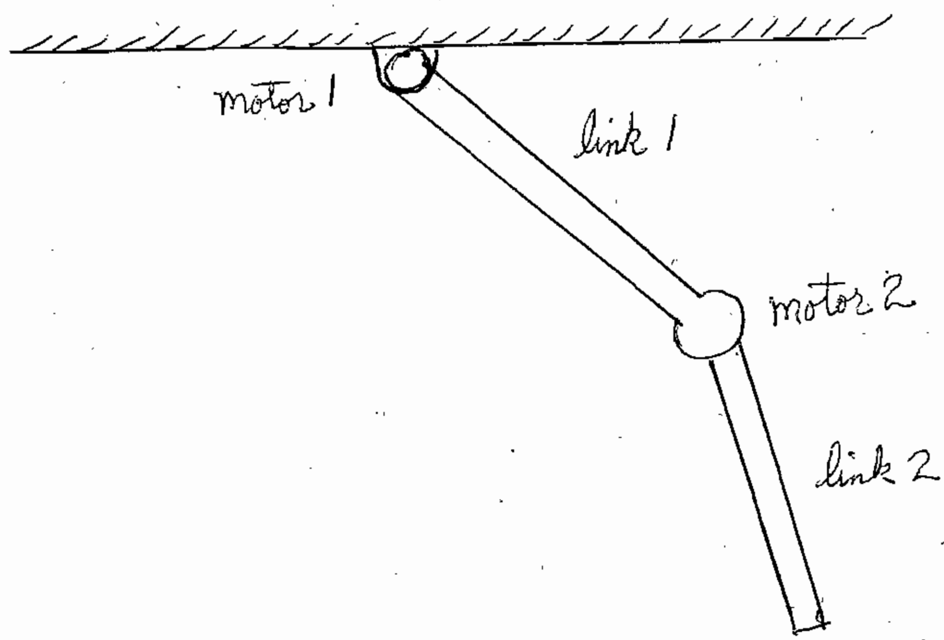
Doctoral Qualifying Examination , January 2004
Mechanical Engineering, Columbia University

1. A particle of mass M is connected by springs of stiffness k to the ends of a smooth tube T, in which it moves. T rotates in the horizontal plane XY about a vertical axis Z with prescribed angular speed Ω. If the springs are unstretched when the mass is at the origin O, obtain the equation of motion of the mass. Find all equilibrium solutions and discuss their stability, for the full range of possible parameter values.



DYNAMICS **Doctoral Qualifying Examination , January 2004**
Mechanical Engineering, Columbia University

2. A two link robot moves in the vertical plane under the influence of gravity. One motor is mounted at the base support for the robot and applies torque to the first or base link. The second motor is mounted on the first link to apply torque to the second link. Derive the equations of motion. Define all symbols you use, all physical parameters you use. Justify how you introduce the torque inputs into the equations.



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VIBRATIONS

Doctoral Qualifying Examination, January 2004
Mechanical Engineering, Columbia University

Problem 1

Consider the following system. All the mass is concentrated in three places; the rest of the rigid bar is massless. $m_1 = m_2 = m_3 = 1$ kg, $k = 1$ N/m, and $l = 1.5$ m.

(a) Using Lagrange method to find the system's equations of motion.

(b) Determine the eigenvectors and natural frequencies.

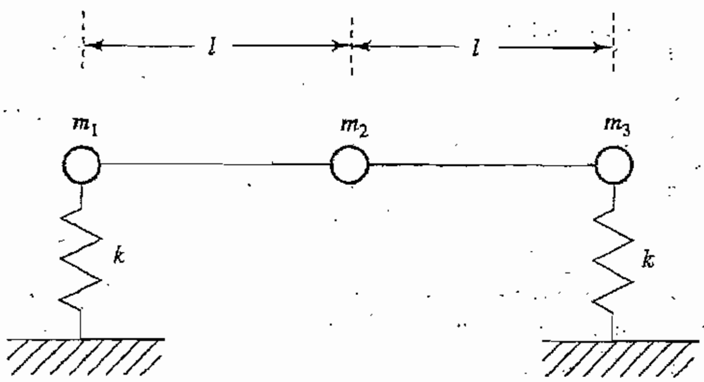

Fig. 1

Problem 2

Calculate the natural frequency equation and mode shape of the longitudinal vibration of a continuous bar which is free at $x = 0$ and connected to ground through a mass and a spring as illustrated in Fig. 2. Also determine the solution for $w_o(x) = \cos \sigma_3 x$ and $\dot{w}_o(x) = 0$, where $\sigma_3 = \omega_3/c$ and $c = \sqrt{E/\rho}$.

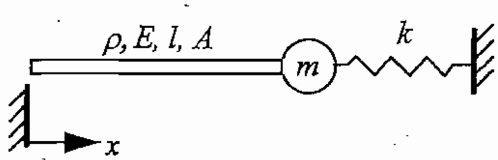

Fig. 2