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CONTROL
Doctoral Qualifying Examination, May 2016
Mechanical Engineering Department, Columbia University

Problem 1

(A) Consider the differential equation

$$\frac{d^2y}{dt^2}+\frac{dy}{dt}+y=\cos(\omega t)$$

and let omega be equal to one.

Find a particular solution using the annihilator method, and explain the procedure and why it works. In other words, find a differential operator that annihilates the right hand side of the equation making the equation homogeneous and then find the solution you need. Explain the logic and explain under what conditions it will work.

(B) The steady state particular solution to the above equation is also the frequency response of the system. Explain how you find frequency response of a system based on its transfer function, and then find the response for the above system.

(C) From the solution in (A) show that it is the same solution as in (B). For more general problems use $\cos(x+y)=\cos(x)\cos(y)-\sin(x)\sin(y)$ to show how solutions in the form obtained in (A) are converted to the form obtained in (B), including explaining how you determining the phase angle quadrant.

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Problem 2

$$\frac{d^2y}{dt^2}+3\frac{dy}{dt}+2y=2u$$

(A) Define state variables used for controllable canonical form and then use them to write this equation in controllable canonical state variable form.
(B) Define state variables to use for observable canonical form, and from the definitions convert the scalar equation to observable canonical form.
(C) Based on the definitions, create a transformation matrix that converts the controllable canonical form state into the observable canonical form state.
(D) What matrix converts from observable canonical form state to controllable canonical form state?
(E) Starting from the controllable canonical form state space equation in part (A), use the transformation matrix to convert the state variables that appear in that equation to those of the observable canonical form, and show that it results in the observable canonical form state space equation in (B).

$$\frac{d^2y}{dt^2}+3\frac{dy}{dt}+2y=4\frac{du}{dt}+2u$$

(F) Convert this new equation into controllable canonical form. Define the state variables.
(G) Define the state variables you use to create the observable canonical form for this new equation, and then produce this state space equation.

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Problem 3

Consider the following matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

(A) What is the characteristic equation for this matrix?

(B) What is the definition of e^{At}.

(C) By substituting the above A into the definition, and computing term by term, find e^{At}.

(D) Consider the following state space system

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

with $C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Define asymptotically stable, marginally stable, unstable. Which category applies to this state space equation?

(E) Find the solution to the state space equation for input $u(t) = 1$ by evaluating the general solution equation

$$y(t) = Ce^{At}x(0) + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau$$

(F) Define what is meant by controllability. Is this system controllable?

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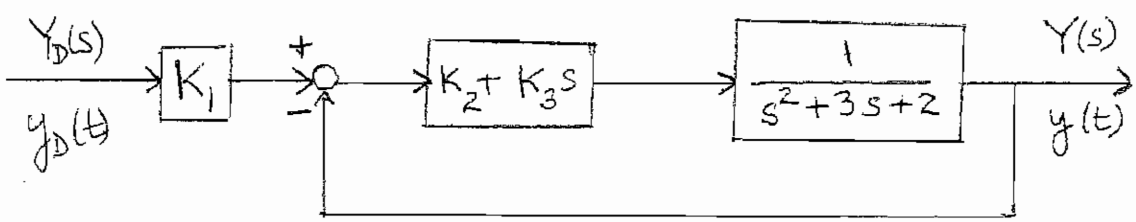
Problem 4

For the block diagram given below:

(A) What is the DC gain of the system, i.e. the constant steady state response to a constant input equal to unity.
(B) Given K_2 and K_3, what value of K_1 will produce zero steady state error to a constant command. Define steady state and error for this problem.
(C) Adjust the gains K_2 and K_3 to place the poles of the closed loop system at $s=-3$ and $s=-4$. Explain your reasoning.
(D) What settling time is associated with this controller design? Define settling time.



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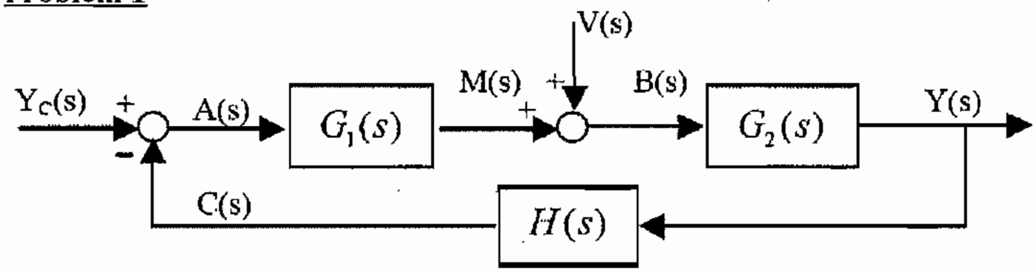
Doctoral Qualifying Examination, January 2015
Mechanical Engineering Department, Columbia University

Problem 1



where $G_1(s) = K \geq 0$, $G_2(s) = \dfrac{1}{s(s+1)}$, $H(s) = 1$.

(A) Define asymptotic stability. For the above system, what range of gains K produce asymptotic stability. Explain your reasoning.

(B) Consider a disturbance $V(s)$ that corresponds to a constant disturbance $v(t) = 1$. What is the effect of this disturbance on the output $y(t)$ after the transients have disappeared, if the feedback controller is turned off?

(C) Now turn on the controller, what is the effect of this disturbance on the output, after the transients have disappeared?

(D) What is the error in response to a constant command $y_c(t) = 1$ after the transients have disappeared?

(E) What is the error in response to a command $y_c(t) = t$ after transients have become negligible?

(F) If both a command $y_c(t) = t$ and a disturbance $v(t) = 1$ are present, after transients are negligible what is the error as a function of time?

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Problem 2

Consider the following system

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$
$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

(A) Find the transfer function from $U(s)$ to $Y(s)$.

(B) Find a scalar ordinary differential equation relating $u(t)$ to output $y(t)$.

(C) Given the initial conditions $x_1(0), x_2(0)$ find the corresponding initial conditions for the scalar ordinary differential equation for $y(t)$ in part (B).

(D) Put the scalar differential equation of part (B) into controllable canonical form, with state variables $\bar{x}_1(t), \bar{x}_2(t)$.

(E) Find a transformation of state variables $\bar{x}(t) = Px(t)$ relating the state variables of the original given differential equation and the state variables used in part (D).

(F) Use this transformation of state variables to convert either the state variable equation in (D) to the given equation above, or vice versa, by substitution.

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Mechanical Engineering Department, Columbia University

Problem 3

One of the root locus rules is the following:

RULE. The root locus exists on the real axis on any interval for which the number of real poles plus real zeros to the right of the interval is an odd number.

Prove that this rule is correct. Consider all possible situations. Explain what your starting point is for your proof, and indicate how one develops this starting point.

Give a detailed development from basic principles.

Problem 4

(A) Define the exponential of square matrix A times t, e^{At}.

(B) Make a mathematically rigorous proof that if A and B are two square matrices of the same dimension, and $AB \neq BA$, then $e^{At}e^{Bt} \neq e^{(A+B)t}$.

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Problem 1

$$\frac{d^2y}{dt^2}+3\frac{dy}{dt}+2y=0$$

(A) Convert this scalar differential equation into state variable controllable canonical form

$$\dot{x}=Ax$$

(B) The solution to this state variable equation is sometimes written as

$$x(t)=e^{At}x(0)$$

By taking the Laplace transform of $\dot{x}=Ax$, find the Laplace transform of the matrix exponential $L(e^{At})$.

(C) Show that the first column of the matrix exponential e^{At} is the solution $x(t)$ to the state variable equation $\dot{x}=Ax$ when the initial state is $x(0)=\begin{bmatrix}1\\0\end{bmatrix}$, and the second column is the solution when $x(0)=\begin{bmatrix}0\\1\end{bmatrix}$.

(D) By solving the original scalar equation and using (C) find e^{At}.

(E) Take the Laplace transform of the result in (D) and check that it matches the $L(e^{At})$ in part (B).

$$L\left(\frac{d^ny(t)}{dt^n}\right)=s^nY(s)-s^{n-1}y(0)-s^{n-2}\frac{dy}{dt}(0)-\cdots-s^0\frac{d^{n-1}y}{dt^{n-1}}(0) \qquad L(e^{-at})=\frac{1}{s+a}$$

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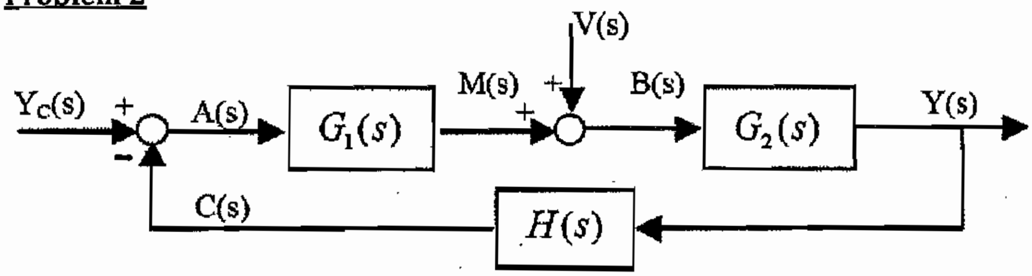
Problem 2

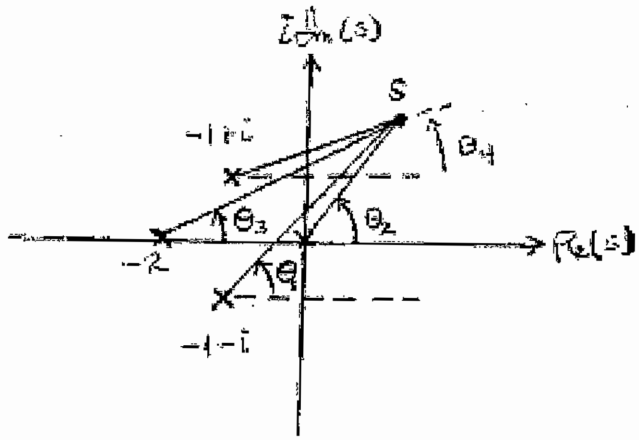


$$G_1(s) = \frac{K}{s} \qquad G_2(s) = \frac{1}{(s+2)[(s+1)^2+1]} \qquad H(s) = 1$$

(A) This problem is to prove from basic principles that a complex number s is a root of the characteristic polynomial of the closed loop system above for some value of gain K, if the angles shown in the diagram given below satisfy the equation

$$\theta_1 + \theta_2 + \theta_3 + \theta_4 = 180^\circ \pm \ell 360^\circ$$

for some $\ell = 1,2,3,\ldots$ Explain every step of the logic needed to establish this result.



(B) Is the point $s = 1 + 2i$ on the root locus? If so what is the gain K that puts a root there.

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Mechanical Engineering Department, Columbia University
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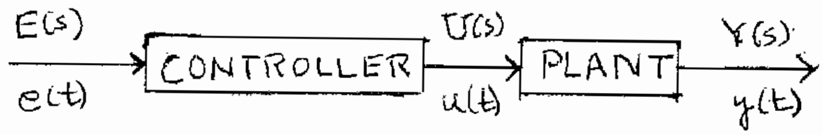
Problem 3



Let the plant be

$$\frac{dy}{dt}+y=u$$

and the controller be

$$\frac{du}{dt}+2u=e$$

(A) Find the transfer function for each of these equations.

(B) Find the transfer function from input $E(s)$ to output $Y(s)$.

(C) Convert this to a differential equation.

(D) Give the general solution of this differential equation when the input $e(t)$ = 0.

(E) Evaluate the arbitrary constants in the general solution in terms of the natural initial conditions for this differential equation.

Now solve the same problem in a different way:

(F) Take the Laplace transform of the plant differential equation, and the Laplace transform of the controller differential equation, but this time do not assume zero initial conditions. Again set $e(t)$=0.

(G) Solve these equations for $Y(s)$. Then take the inverse transform.

(H) By comparing the solution from (G) to the solution from (E), write the equivalence between the two sets of initial conditions.

If the plant and the controller are two pieces of hardware, which set of initial conditions are you most likely to know physically?

$$L\left(\frac{d^n y(t)}{dt^n}\right)=s^n Y(s)-s^{n-1}y(0)-s^{n-2}\frac{dy}{dt}(0)-\cdots-s^0\frac{d^{n-1}y}{dt^{n-1}}(0) \qquad L(e^{-at})=\frac{1}{s+a}$$

Doctoral Qualifying Examination, January 2014
Mechanical Engineering Department, Columbia University
CONTROL

Problem 1

Consider the following Multi-Input Multi-Output system

$$\frac{d^2y_1}{dt^2} + 3\frac{dy_1}{dt} + 2(y_1 - y_2) = u_1 + u_2$$

$$\frac{dy_2}{dt} + 3(y_2 - y_1) = u_2$$

Let the Laplace transforms of the input vector and of the output vector be

$$U(s) = \begin{bmatrix} U_1(s) \\ U_2(s) \end{bmatrix} \qquad Y(s) = \begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix}$$

(A) Find the transfer function matrix $G(s)$ such that

$$Y(s) = G(s)U(s)$$

(B) What is the characteristic polynomial that determines if this system is asymptotically stable? Is it asymptotically stable?

Problem 2

Consider the exponential of a matrix e^{At} where A is a square matrix that is diagonalizable, and t is time.

(1) Define the exponential of a matrix.

(2) What conditions does matrix A have to satisfy so that it can be diagonalized?

(3) Show that for any invertible matrix M

$$e^{M^{-1}AMt} = M^{-1}e^{At}M$$

(4) If M is chosen to diagonalize matrix A, create a way to evaluate the exponential of matrix A, e^{At}, in terms of exponentials of scalars.

Problem 3

(1) Define the unit step response of a control system.

(2) Consider

$$\frac{d^2y}{dt^2}+3\frac{dy}{dt}+2y=1.8u$$

Find the unit step response.

(3) What is the steady state error in this response?

(4) Give a definition of settling time. What is the settling time of this system? What are the time constants of this system?

(5) Is the system underdamped, critically damped, or overdamped? Explain why?

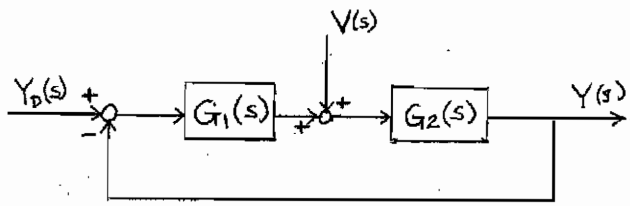
(6) Consider a related system with a zero introduced

$$\frac{d^2y}{dt^2}+3\frac{dy}{dt}+2y=\frac{du}{dt}+1.8u$$

Explain all methods that you can think of that you could use to find the unit step response for this new system.

Doctoral Qualifying Examination, January 2013
Mechanical Engineering, Columbia University
CONTROL

Problem 1



$G_1(s)$ is the controller, $G_2(s)$ is the plant, $Y_D(s)$ is the command, $V(s)$ is the disturbance, and $Y(s)$ is the control system response.

(1) Generally, one expects that proportional control systems have constant errors after transients have become negligible, when given a constant command, and one expects that the influence of a constant disturbance on the output produces constant error.

(2) Genearally one expects that a proportional control system will have linearly increasing error in response to a linearly increasing disturbance.

(3) An integral controller is expected to have zero error for these inputs, but constant error for linearly increasing commands and linearly increasing disturbances.

Questions:

(A) Prove that (1) is true, specifically explaining what assumptions you have to make about the plant in order for the system to have this property (1).

(B) Do the same for item (2).

(C) Do the same for item (3).

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Problem 2

(A) Define what is meant by the exponential of a square matrix F.

(B) Suppose matrix F is a function of time t. Find a formula for the derivative of the matrix exponential of $F(t)$ with respect to time t. Explain all steps.

(C) Consider the time varying linear state variable differential equation

$$\dot{x}(t) = A(t)x(t)$$

The solution of the scalar version of this equation suggests that the following might be the solution to this time varying matrix equation Show under what conditions

$$x(t) = \exp\left(\int_0^t A(\tau)d\tau\right)x(0)$$

Examine under what conditions it is the solution satisfying initial conditions $x(0)$ at time $t = 0$.

(D) Generalize the above result to go from an initial condition at time t_1 to a future time t_2.

(E) Then relate the 2 time varying state transition functions that go from t_1 to t_2, and go from t_2 to t_3, to the state transition function that goes from t_1 to t_3. Thus establish mathematically one of the properties of these exponential solutions.

Doctoral Qualifying Examination, January 2013
Mechanical Engineering, Columbia University
CONTROL

Problem 3
A standard form for under-damped second order differential equations is

$$\frac{d^2y}{dt^2} + 2\zeta\omega_n \frac{dy}{dt} + \omega_n^2 y = 0$$

(A) What range of ζ correspond to under-damped systems? What does under-damped mean?

(B) What are the root locations for the associated characteristic polynomial (under damped case).

(C) What is the radial distance to the roots from the origin of the complex plane (under-damped case).

(E) What is the angle made between the negative real axis, and the line from the origin to the roots? Relate it to the coefficients in the differential equation.

CONTROL SYSTEMS
Doctoral Qualifying Examination, January 2012
Mechanical Engineering, Columbia University

Problem 1

Consider the following system

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

(A) Find the transfer function from $U(s)$ to $Y(s)$.

(B) Find a scalar ordinary differential equation relating $u(t)$ to output $y(t)$.

(C) Given the initial conditions $x_1(0), x_2(0)$ find the corresponding initial conditions for the scalar ordinary differential equation for $y(t)$ in part (B).

(D) Put the scalar differential equation of part (B) into controllable canonical form, with state variables $\bar{x}_1(t), \bar{x}_2(t)$.

(E) Find a transformation of state variables $\bar{x}(t) = Px(t)$ relating the state variables of the original given differential equation and the state variables used in part (D).

(F) Use this transformation of state variables to convert either the state variable equation in (D) to the given equation above, or vice versa, by substitution.

CONTROL SYSTEMS
Doctoral Qualifying Examination, January 2012
Mechanical Engineering, Columbia University

Problem 2

For the following system

$$\dot{x} = \begin{bmatrix} -3 & 1 & 0 \\ 0 & 0 & -2 \\ -4 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 2 \\ 1 & 0 \\ 0 & 3 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x$$

(a) Is this system asymptotically stable, marginally stable, or unstable? Explain your reasoning.

(b) Is the system observable? Show your reasoning.

(c) Is the system controllable? Show your reasoning.

(d) Is the system controllable using only the second component of the input vector u?

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CONTROL SYSTEMS
Doctoral Qualifying Examination, January 2012
Mechanical Engineering, Columbia University

Problem 3

Consider the following system

$$\dot{x}(t) = Ax(t) \quad ; \quad \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

(A) Write the two scalar equations associated with the two components of the state $x(t)$. Then solve these two scalar equations successively, starting with the uncoupled one.

(B) The state transition matrix $\Phi(t) = e^{At}$ is a matrix satisfying

$$x(t) = \Phi(t)x(0)$$

Show that if you know the solution for $x(t)$ for the initial condition $[1 \quad 0]^T$, and you know the solution for the initial condition $[0 \quad 1]^T$ (superscript T for transpose), then you know $\Phi(t)$.

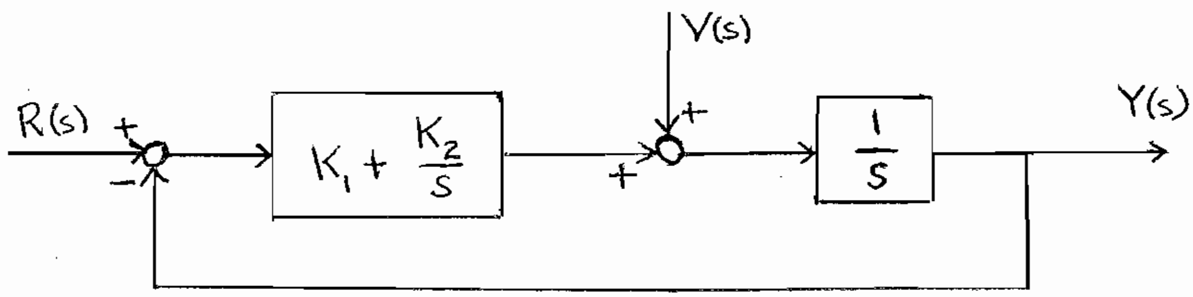
(C) Use part (B) to find e^{At} for the given system.

Doctoral Qualifying Examination, January 2011
Mechanical Engineering, Columbia University

CONTROL SYSTEMS

Problem 1

Consider the following feedback control system



(A) Consider disturbances $v(t)$ that are of the form 1, t, t^2, ... Determine the steady state error (i.e., the error that does not contain any terms that are solutions of the homogeneous equation) in the output of the control system associated with these different forms of disturbances.

(B) Do the same for commands $r(t)$.

(C) What constraints do the gains K_1, K_2 have to satisfy in order for the feedback control system to be asymptotically stable?

Problem 2

Convert the following state variable differential equation to a second order scalar differential equation for $y(t)$:

$$\dot{x}(t) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} x(t) + u(t)$$

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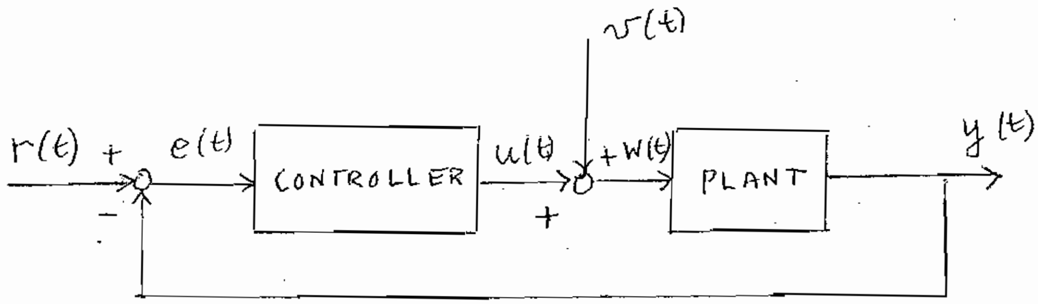
Doctoral Qualifying Examination, January 2011
Mechanical Engineering, Columbia University

CONTROL SYSTEMS

Problem 3



Controller:
$$\dot{x}_c(t) = A_c x_c(t) + B_c e(t)$$
$$u_c(t) = C_c x_c(t)$$

Plant:
$$\dot{x}_p(t) = A_p x_p(t) + B_p w(t)$$
$$y(t) = C_p x_p(t)$$

Find a state space model of the closed loop control system above with two inputs -- the command r(t) and the disturbance v(t) – and output y(t).

CONTROL

Doctoral Qualifying Examination, January 2010
Mechanical Engineering Department

Problem 1

Consider the following system

$$\dot{x}(t) = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$$
$$y(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} x(t)$$

(A) Use $x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$ to find $y(t)$ when $u(t) = 1$

(B) What is the unit step response of this system?

(C) Find the transfer function of the given state space system, relating the transform of the input to the transform of the output. Then write a scalar differential equation that is equivalent to the given state space matrix differential equation.

Problem 2

Consider a system governed by

$$\frac{d^2y(t)}{dt^2} + \frac{dy(t)}{dt} + y(t) = u(t)$$

(A) Design a state (x) feedback controller $u = \begin{bmatrix} K_1 & K_2 \end{bmatrix} x$ by pole placement putting the poles at -5 and -10.

(B) Add a term to this state feedback controller $u = \begin{bmatrix} K_1 & K_2 \end{bmatrix} x + K_3 y_D(t)$ that includes a gain times the command or desired output $y_D(t)$. Adjust this constant to produce a DC gain of unity.

(C) Design an observer by pole placement that places the poles at -20 and -30, when only $y(t)$ is measured.

(D) Most likely the resulting controller that uses the observer to supply the state needed in the control law will be implemented in a digital computer or microprocessor. What formulas does this computer have to solve in real time?

CONTROL

Doctoral Qualifying Examination, January 2010
Mechanical Engineering Department

Problem 3

(A) Graph the asymptotic form (straight line approximation form) of the 20log magnitude vs. frequency Bode plot for the following transfer function

$$T(s) = \frac{100(s+1)}{(s+10)(s+100)}$$

Explain all of your reasoning, stating what information you are using to obtain your result and how you are using it.

(B) Graph the straight line approximation of phase Bode plot for this transfer function. Again, say in detail what information you are using and how you are using it to obtain your results.

(C) If these plots were plots of the open loop transfer function of some feedback control system, show how you determine the approximate gain and phase margins of the feedback system from the plots. Define these and then give your reasoning. Does the result predict stability or instability?

Doctoral Qualifying Examination, January 2009
Mechanical Engineering, Columbia University

CONTROL

Problem 1:

Prove that if

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 0 & 0 & 5 & 6 \\ 0 & 0 & 7 & 8 \end{bmatrix}$$

then $e^{At} = \begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix}$ where the 0 entries are two by two zero matrices, and

$$E_1 = e^{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} t} \; ; \; E_2 = e^{\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} t}$$

Show and explain all of your logic.

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CONTROL

Problem 2:

One most often uses state variable equations in controllable canonical form, e.g. with the system matrix in the form

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix}$$

Let the eigenvalues of this matrix be $\lambda_1,\ \lambda_2,\ \lambda_3$. Show that the Vandermonde matrix

$$V = \begin{bmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{bmatrix}$$

allows one to diagonalize matrix A by a similarity transformation

$$V^{-1}AV = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

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CONTROL

Problem 3:

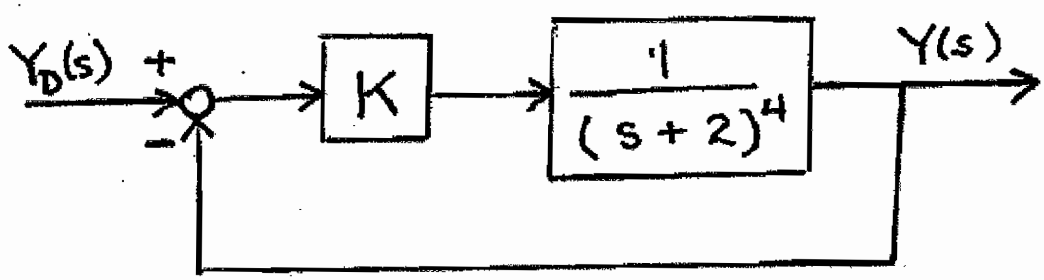
For the proportional control system shown below:

a) What range of controller gains K (both positive and negative) makes this feedback control system asymptotically stable? What gains produce marginal stability?

b) When you increase the gain K to the highest value for marginal stability, use Routh to know what the roots are that are on the imaginary axis. By dividing these roots into the polynomial for this gain, find the other roots analytically.

c) Find the angles of departure for the roots from the poles at -2 as the gain K is increased from zero.

d) Now consider changing the sign of the gain K to be negative. Determine the angles of departure from -2 as the gain K starts at zero and decreases to negative numbers. Explain how you make your conclusions.



CONTROLS

Doctoral Qualifying Examination, January 2008
Mechanical Engineering, Columbia University

Problem 1

Consider the following two-input, two-output system:

$$\ddot{y}_1 + 3\dot{y}_1 + 2(y_1 - y_2) = u_1 + u_2$$
$$\dot{y}_2 + 3y_2 - y_1 = u_2 - 2u_1$$

(A) Define what is meant by state variables for ordinary differential equations.
(B) Pick an appropriate set of state variables for the above coupled equation set.
(C) Write the system in coupled first order state variable form with the usual system dynamics equation and the output equation.
(D) By taking the Laplace transform of the original equations, find the transfer function from u_1 to y_2.

Problem 2

This problem develops an unusual way to diagonalize a matrix.

(A) Start with the transfer function

$$Y(s) = G(s)U(s) = \left[\frac{1}{(s+1)(s+2)}\right]U(s)$$

and convert this to controllable canonical form. Record the definitions you used for the state variables, and denote the state variable column vector by $x(t)$.

(B) Make a partial fraction expansion of the transfer function, writing it as a sum of two first degree terms. Define the product of the first of these terms with $U(s)$ as the transform of differently defined state variable $\overline{X}_1(s)$, and let the second term similarly define the transform of a second state variable $\overline{X}_2(s)$. Convert these equations back to the time domain, and then write the two equations in state variable form – which will be diagonal form. Denote this two dimensional state vector in the time domain by $\bar{x}(t)$.

(C) Now, by looking at the definition used for the state variables $x(t)$ in (A), and the $\bar{x}(t)$ used in (B), find a 2 by 2 transformation matrix that converts one to the other.

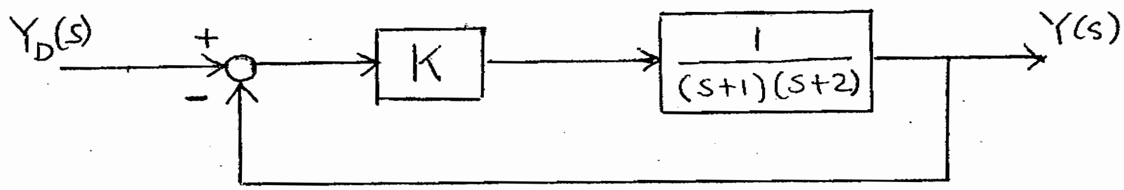
(D) Show that by using this relationship between the two state vectors, you can convert the controllable canonical form state variable equations to diagonal form.

CONTROLS

Doctoral Qualifying Examination, January 2008
Mechanical Engineering, Columbia University

Problem 3

Consider the following proportional control system



(A) Find the closed loop characteristic equation.
(B) Solve this equation analytically for all positive values of control gain *K*. Then plot the possible root locations in the complex plane for all positive *K*.
(C) Now apply all of the root locus rules to this problem, stating the rule and showing what it indicates about the root locus plot.
(D) Can you use root locus methods to determine the angle of departure from the real axis, and show that it is consistent with the result in (B)?

CONTROL

Doctoral Qualifying Examination, January 2007
Mechanical Engineering, Columbia University

Problem 1

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -K & -2 & -3 & -1 \end{bmatrix} \quad ; \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 5 \end{bmatrix} \quad ; \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$$

(A) Find if this system is asymptotically stable for any value of controller parameter K, and if so what is the range of values of K producing this.

(B) You would like all transients to decay at least as fast as e^{-t}. Find the range of K satisfying this condition, if such a range exists.

Problem 2

Solve the following state variable differential equation by the following two methods

$$\dot{x}(t) = \begin{bmatrix} 0 & +\omega \\ -\omega & 0 \end{bmatrix} x(t) \quad ; \quad x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(A) Method One: Substitute a solution of the form $x(t) = ve^{st}$ and determine the vector v and the scalar s to satisfy the equation. Create the general solution by superposition, and then evaluate the arbitrary constants to satisfy the initial conditions.

(B) Method Two: Apply Laplace transforms to the matrix differential equation and then solve.

[note: $L[dy(t)/dt] = sY(s) - y(0)$, $e^{i\theta} = \cos\theta + i\sin\theta$, and there is a Laplace transform table in the Dynamics and Vibrations part of this exam]

CONTROL

Doctoral Qualifying Examination, January 2007
Mechanical Engineering, Columbia University

Problem 3

Consider a unity feedback control system with transfer function in the forward part of the loop (controller times plant) given by

$$G(s) = \frac{K}{s(s^2 + 6s + 13)}$$

where K is the gain in the proportional controller.

(A) Draw a root locus plot for the path of the roots as K is increased from zero to infinity. Explicitly state what rules you are using to determine the form of the graph.

(B) Find the largest gain K for which the system is just on the boundary of going unstable, and find the roots that are on the imaginary axis.

(C) Find the remaining root to the closed loop characteristic polynomial for this gain.

(C) For this gain, what is the steady state response to a command of one? Then, find the unit step response of this system for this gain.

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CONTROL

Doctoral Qualifying Examination, January 2006
Mechanical Engineering, Columbia University

Problem 1

Consider the following state space equation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

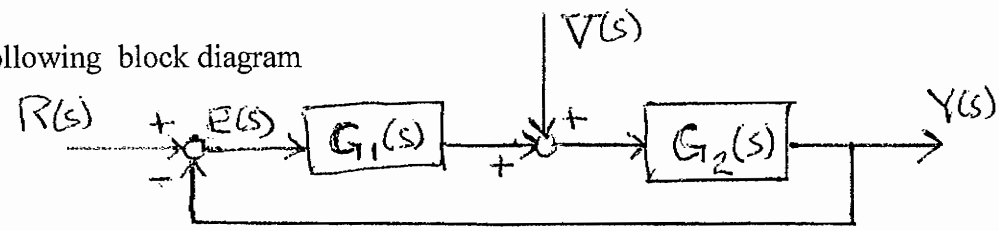
(A) Is this system asymptotically stable, marginally stable, or unstable? Explain your answer.
(B) Is this system controllable? Explain your answer.
(C) Is this system observable? Explain your answer.
(D) Find the transfer function from $U(s)$ to $Y(s)$. Explain your method.
(E) Write a scalar second order differential equation relating input $u(t)$ to output $y(t)$.
(F) Convert this scalar second order differential equation into state space form, using any method you desire. Define the state variables that you are using.
(G) Can you relate the state variables you used in (D) to those in the original equation above?

CONTROL

Doctoral Qualifying Examination, January 2006
Mechanical Engineering, Columbia University

Problem 2:

In the following block diagram



The controller and the plant are

$$G_1(s) = K/s^m$$
$$G_2(s) = \frac{1}{(s+1)(s+2)}$$

where m is an integer.

(A) Write a differential equation for the error $e(t) = r(t) - y(t)$ when there is both a command $r(t)$ and a disturbance $v(t)$.
(B) By finding particular solutions to this equation, give the steady state error or the rate of growth of the error for constant commands, linearly increasing commands, accelerating commands, e.g. $r(t) = 1,\ t,\ t^2$. Do this for $m = 0,\ 1,\ 2,\ \ldots$
(C) Do the same for constant, linearly increasing, or accelerating disturbances.
(D) Discuss what influence using a large m has on the solution of the homogeneous equation. Can you adjust K to give good transient response, when m is large?
(E) Is it a good idea to use a double integral controller? Explain your answer.

CONTROL

Doctoral Qualifying Examination, January 2006
Mechanical Engineering, Columbia University

Problem 3:

Convert the following scalar differential equations

(A) $$\frac{d^3y(t)}{dt^3}+3\frac{d^2y(t)}{dt^2}+2\frac{dy(t)}{dt}+y(t)=\frac{d^2u(t)}{dt^2}+5\frac{du(t)}{dt}$$

(B) $$\frac{d^3y(t)}{dt^3}+3\frac{d^2y(t)}{dt^2}+2\frac{dy(t)}{dt}+y(t)=4\frac{d^3u(t)}{dt^3}+\frac{d^2u(t)}{dt^2}+5\frac{du(t)}{dt}$$

into state variable form

$$\dot{x}=Ax+Bu$$
$$y=Cx+Du$$

Clearly explain your definitions of state variables, and how you get the results you give.

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CONTROL

Doctoral Qualifying Examination, January 2005
Mechanical Engineering, Columbia University

Problem 1

For the following differential equation

$$\frac{d^3y}{dt^3}+2\frac{d^2y}{dt^2}+3\frac{dy}{dt}+4y=7u(t)$$

(A) Write the equation in controllable canonical form. Be specific as to how you define each of the state variables in terms of $y(t)$. Use a subscript c on the state and the matrices to identify them as related to the controllable canonical form.

(B) Write the equation in observable canonical form. Again, be specific as to how you define each of the state variables in terms of $y(t)$. This time use a subscript o on the state and the matrices to identify them as related to the observable canonical form.

(C) Using the definitions of the state variables in terms of $y(t)$ that you used in each case, create a matrix that when multiplied times the controllable canonical form state produces the observable canonical form state (or vice versa if you wish).

(D) Use this transformation of state variables from (C) to convert the state equation and output equation for the controllable canonical form into the state equation and output equation for the observable canonical form (or vice versa if you wish).

Problem 2

For the following system

$$\frac{d}{dt}\begin{bmatrix}x_1(t)\\x_2(t)\end{bmatrix}=\begin{bmatrix}1&2\\3&4\end{bmatrix}\begin{bmatrix}x_1(t)\\x_2(t)\end{bmatrix}+\begin{bmatrix}0\\1\end{bmatrix}u(t)$$

$$y(t)=\begin{bmatrix}1&0\end{bmatrix}\begin{bmatrix}x_1(t)\\x_2(t)\end{bmatrix}+5u(t)$$

(A) Find the transfer function relating the Laplace transform of the input $U(s)$ to the transform of the output $Y(s)$.

(B) Find a scalar differential equation relating $u(t)$ to the output $y(t)$.

(C) Is this system controllable?

(D) Is this system observable?

(E) Is this system asymptotically stable?

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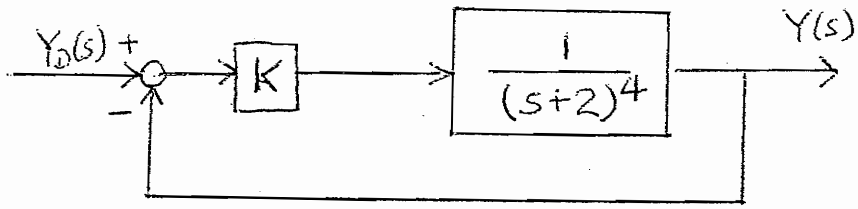
CONTROL

Doctoral Qualifying Examination, January 2005
Mechanical Engineering, Columbia University

Problem 3



(A) Draw the root locus plot for the above system for positive K. Include all the normal items if they apply to the problem, including:

- location of poles and zeros
- where the locus exists on the real exis
- centroid of the asymptotes
- angles of the asymptotes
- angle of departure and angle of arrival
- break away points from the real axis
- where the roots cross the imaginary axis

(B) What range of gains K causes the system to have a settling time of 4 seconds or less? Use Routh criterion to find this, and then use the criterion to find the location of the dominant roots for the largest gain in this range.

CONTROL

Doctoral Qualifying Examination , January 2004
Mechanical Engineering, Columbia University

Problem 1

Consider the following control system

$$Y(s) = G(s)Y_d(s)$$
$$G(s) = \frac{1}{s^2 + s + 1}$$

where $Y_d(s)$ is the command.

(a) Find the unit step response of the system.

(b) Give the damping ratio and the damped natural frequency.

(c) Find the steady state error.

(d) Find the peak overshoot and the time of the peak overshoot.

(e) Find the size of the next maximum after the peak overshoot as time progresses, and then write a relationship the between the sizes of successive peaks.

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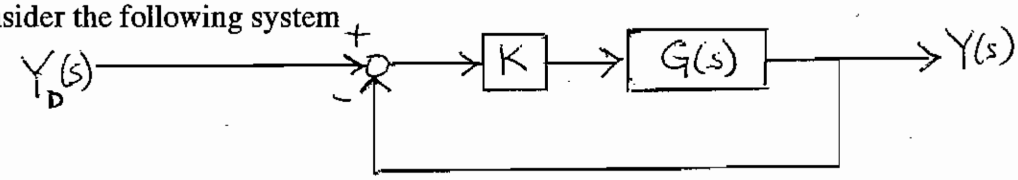
CONTROL

Doctoral Qualifying Examination , January 2004
Mechanical Engineering, Columbia University

Problem 2

Consider the following system



where $G(s) = \dfrac{1}{s(s+2)(s^2+2s+2)}$. Consider all the items that apply to the root locus for this system:

(a) where the locus exists on the real axis
(b) what the asymptotic directions are
(c) what the centroid of the asymptotes
(d) breakaway or breakin points for the real axis
(e) angle of departure or arrival at complex poles or zeros
(f) where the locus crosses the imaginary axis

Also,

(g) what it the gain when the root locus breaks away from the real exis?
(h) what angles does it depart from the real axis?
(i) From the above, sketch what you expect the locus to look like
(j) and then prove that your sketch is correct.

CONTROL

Doctoral Qualifying Examination , January 2004
Mechanical Engineering, Columbia University

Problem 3

(a) Prove that if the input to the following equation is through a zero order hold with sample time T,

$$\dot{x}(t) = Ax(t) + Bu(t)$$

then the state at the sample times $kT,\ k = 0,1,2,....$ can be found from the sampled input $u(kT)$ from a difference equation

$$x((k+1)T) = A_d x(kT) + B_d u(kT)$$

Derive the equations for A_d, B_d.

(b) Prove that

$$\begin{bmatrix} A_d & B_d \\ 0 & I \end{bmatrix} = \exp\left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} T\right)$$

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CONTROL

Doctoral Qualifying Examination , January 2004
Mechanical Engineering, Columbia University

Problem 4

(a) Given a transfer function

$$G(s) = \frac{\overline{K}(s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0)}{(s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0)} \qquad n \geq m$$

If the input to this transfer function is $\cos(\omega t)$, what is the phase lag in the output in the limit as the frequency gets large, as a function of m and n?

(b) Now suppose that the $\cos(\omega t)$ input goes through a zero order hold before going into the transfer function, with a sample time of T seconds in the hold, and we look at the output at the sample times. What is the highest frequency one can have without going above Nyquist? What can you say about the possible phase lag values at this highest frequency in the discrete time system?

(c) If you have a plot of the phase angle of the discrete time system from zero to Nyquist, explain what the plot would look like if you extended it to all frequencies of input, going to infinity. Give a complete justification of you statement.

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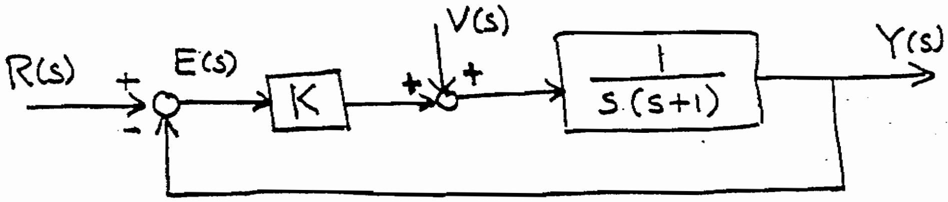
CONTROL

JAN. 2002

PROBLEM 1

Consider the following system for controlling a robot link using a proportional controller:



A. Find E(s) as a function of of the command R(s) and the disturbance V(s). Create the corresponding differential equation whose solution is the error e(t).

B. Find the error after the transients have died away (i.e. after the solution to the homogeneous equation has become zero), by finding the associated particular solutions for the following cases (using the method in the handout)

(i) Constant command r(t)=1 (v(t)=0).
(ii) Constant disturbance v(t)=1 (r(t)=0).
(iii) Ramp command r(t)=t (v(t)=0).
(vi) Ramp disturbance v(t)=t (r(t)=0).

C. Comment on how these results relate the results that are normally associated with a proportional controller. Explain any difference.

D. Find the general solution for the homogeneous equation (i.e. the transient part of the system response) for all positive controller gains K. Give the result in terms of real valued functions only (use the information in the handout.) What range of gains results in pure exponential decay of the transients? What range of K results in decaying oscillatory transients? What value of K results in a repeated root solution?

CONTROL

JAN. 2002

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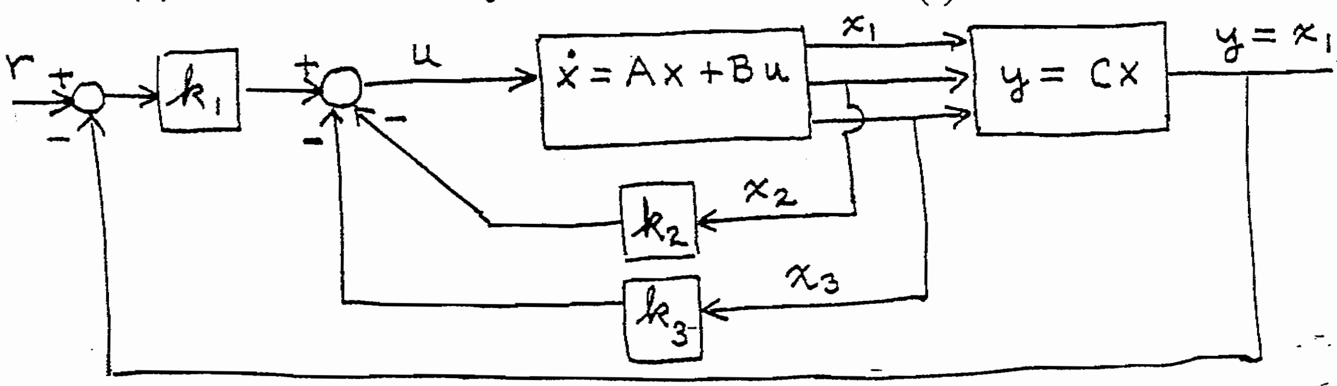
Problem 1

In the following block diagram, matrices A, B, C are given as

$$A=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & -5 & -6\end{bmatrix},\ B=\begin{bmatrix}0\\0\\1\end{bmatrix},\ C=\begin{bmatrix}1 & 0 & 0\end{bmatrix}$$

(a) Determine the feedback gains k_1, k_2, k_3 such that the closed loop poles are located at $s=-2+4i$, $s=-2-4i$, and $s=-10$.

(b) What is the steady state error to a command $r(t)=1$?



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CONTROL

JAN, 2002

Problem 3:

Give the general solution of the following difference equations (general solution of the homogeneous equation plus a particular solution, when needed). Give the answer in terms of real valued functions (no complex numbers involved).

A) $y((k+2)T)+7y((k+1)T)+12y(kT)=1$

B) $y((k+1)T)+y(kT)=kT$

C) $y((k+1)T)+y(kT)=2^k$

D) $y((k+2)T)+2y((k+1)T)+y(kT)=0$

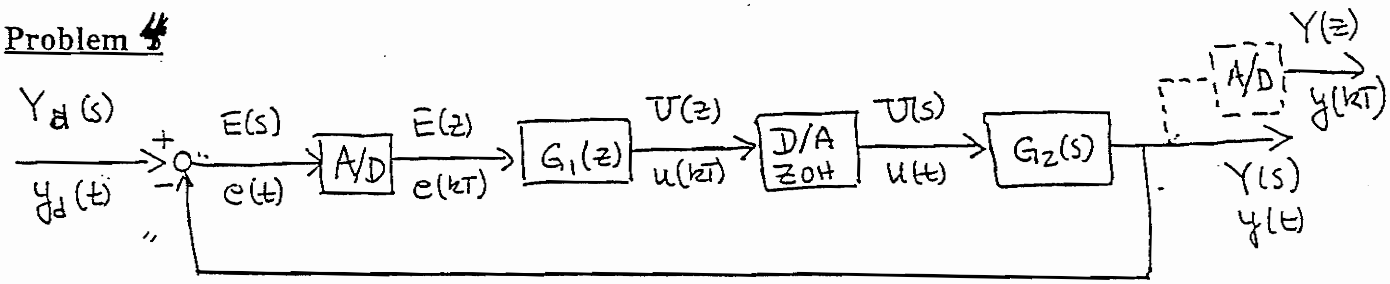
E) $y((k+2)T)+y((k+1)T)+y(kT)=0$

F) Give the solution to (A) above that satisfies $y(kT)=0$ for $k=0$ and for $k=-1$ (i.e., find the unit step response of the system).

CONTROL

JAN. 2002

Problem 4



Let $G_1(z) = \dfrac{z}{z^2 + z + 1}$ and $G_2(s) = \dfrac{1}{s+1}$.

A) Write $G_2(s)$ as a scalar differential equation. Convert to modern state variable form

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$

B) Use state variable results to convert this to a state variable difference equation using the fact that the input to the equation comes from a zero order hold

$$x((k+1)T) = A_D x(kT) + B_D u(kT)$$
$$y(kT) = Cx(kT)$$

C) Write $G_1(z)$ as a scalar difference equation. Then convert it to a state space form like that in Part B by defining appropriate state variables.

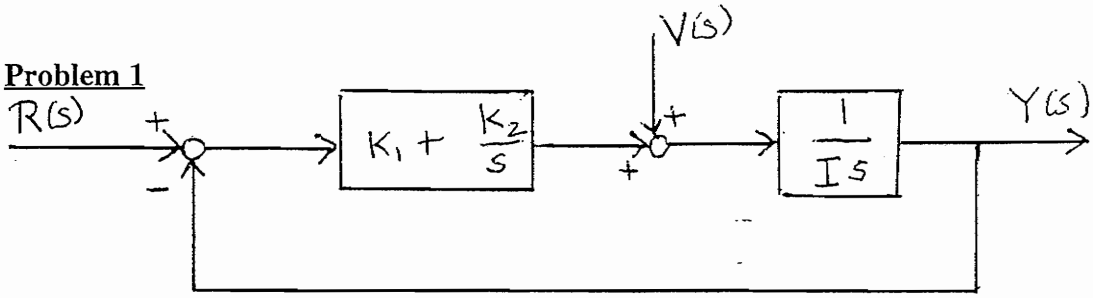
D) Combine the state variable forms from above to give a state variable difference equation representing the closed loop system input/output behavior

$$x_{CL}((k+1)T) = A_{CL} x_{CL}(kT) + B_{CL} y_d(kT)$$
$$y(kT) = C_{CL} x_{CL}(kT)$$

E) Write an expression for the characteristic equation of the closed loop system.

Mechanical Engineering Department, Doctoral Qualifying Examination, January 2001
CONTROL

Problem 1

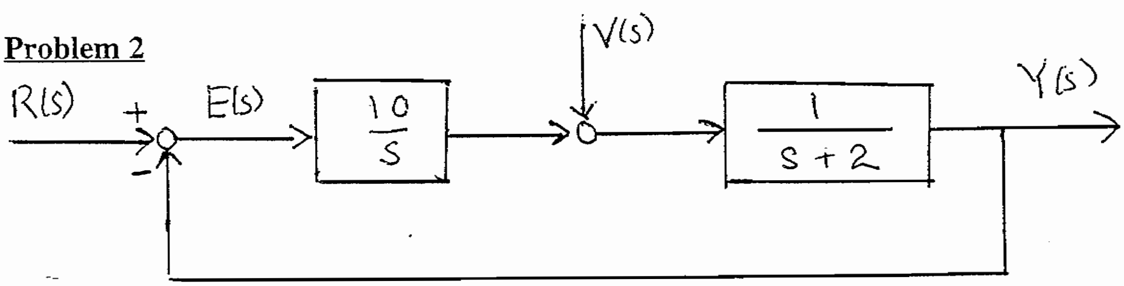


(A) Investigate the response of the above system to step, ramp, t^2, etc. commands $r(t)$ to determine which the system can follow with zero steady state error, or follow with constant steady state error, or follow with linearly increasing error after transients are gone. Do this by creating the differential equation relating commands to their effect on the output, and then solving for the particular solution that contains no transients.

(B) Repeat part (A) except do it for error produced by disturbances $v(t)$. Comment on how your result to this part and part (A) compare to the usual results for proportional or integral controllers, and explain any differences.

(C) What gains are required in the controller in order for the feedback control system to be stable?

Problem 2



(A) In the above system there is a periodic disturbance $v(t) = \cos(t)$. Find the error produced in the output as a result of the disturbance, after all transients have decayed to zero.

(B) Suppose that the disturbance changed to $v(t) = 10\sin(t + \frac{\pi}{4})$. What would be the steady state error that results? Justify your answer.

Mechanical Engineering Department, Doctoral Qualifying Examination, January 2001
CONTROL

Problem 3

Consider the following difference equation

$$y(k+1)+5y(k)+6y(k-1)=10u(k)$$
$$y(-1)=0 \quad y(0)=1$$
$$u(k)=1 \ \forall \ k$$

(A) Solve this equation recursively the way a feedback controller would, i.e. at time step k, use the present and old values of y and u to get the next value. Solve for $y(k)$ for $k=1,2,3,4$.

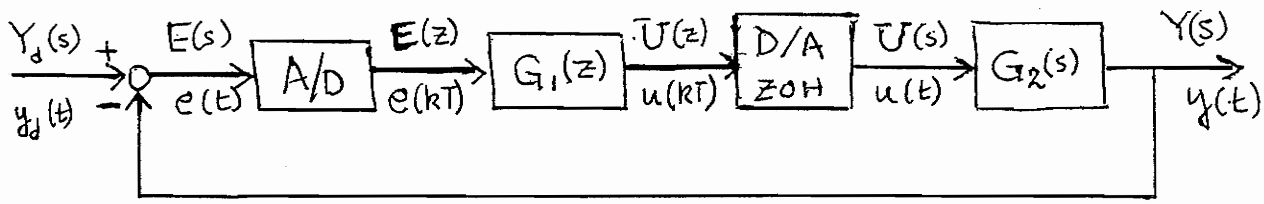
(B) Solve this equation in a manner analogous to solving a differential equation, i.e. find the general solution to the homogeneous difference equation for all k including arbitrary constants, solve for a particular solution for all k, evaluate the constants in terms of the initial conditions.

(C) Solve the difference equation by using z-transforms, including the initial conditions. Note that:

$$Z\{e^{-at}\}=Z\{e^{-akT}\}=\frac{z}{z-e^{aT}}$$
$$Z\{y((k+1)T)=zY(z)-zy(0)$$
$$Z\{y((k+2)T)=z^2Y(z)-z^2y(0)-zy(1)$$

Mechanical Engineering Department, Doctoral Qualifying Examination, January 2001
CONTROL

Problem 4



Let $G_1(z) = \dfrac{z}{z^2 + z + 1}$ and $G_2(s) = \dfrac{1}{s+1}$.

(A) Write $G_2(s)$ as a scalar differential equation. Convert it to state variable form

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$

(B) Use state variable results to convert this to a state variable difference equation

$$x((k+1)T) = A_d x(kT) + B_d u(kT)$$
$$y(kT) = C_d x(kT)$$

(C) Write $G_1(z)$ as a scalar difference equation with appropriate names for the input variable and the output variable.

(D) Convert the scalar difference equation in (C) into a state variable difference equation of the same general form as in (B).

(E) Combine the state variable difference equations for the plant in (B) and for the controller in (D) into one state variable difference equation, with the input being the command to the control system at the sample times, and the output being the response of the control system at the sample times.

MECHANICAL ENGINEERING
QUALIFYING EXAMINATION

January 2000

Control

PROBLEM I

A proportional control system has closed loop transfer function given by

$$R(s) \longrightarrow \boxed{\dfrac{K}{s(s+1)(s+2)+K}} \longrightarrow Y(s)$$

A. Use Routh to find the range of K for which the system is (asymptotically) stable.

B. By shifting the imaginary axis and then using Routh, find the range of K such that all roots of the characteristic equation lie to the left of the vertical line (parallel to the imaginary axis) going through the point s=-0.25.

C. What bound on the time constants of the transients of this control system applies for this range of K in part B?

D. What bound on the settling time is associated with this range of K in part B?

E. Use Routh to determine what values of K put roots on this line, and for these values of K use Routh rules to actually find the roots on this line.

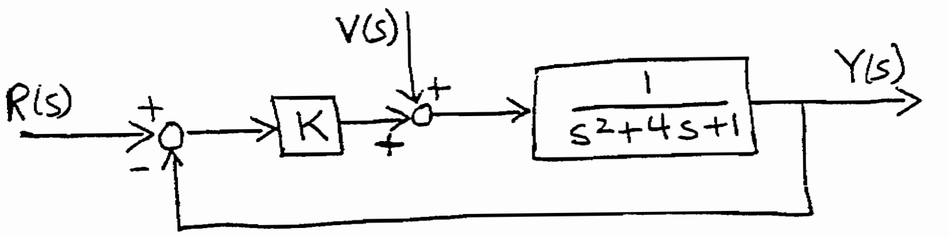
F. What damping ratio is associated with the dominat roots for the gains in E?

MECHANICAL ENGINEERING
QUALIFYING EXAMINATION

January 2000

Control

PROBLEM 2



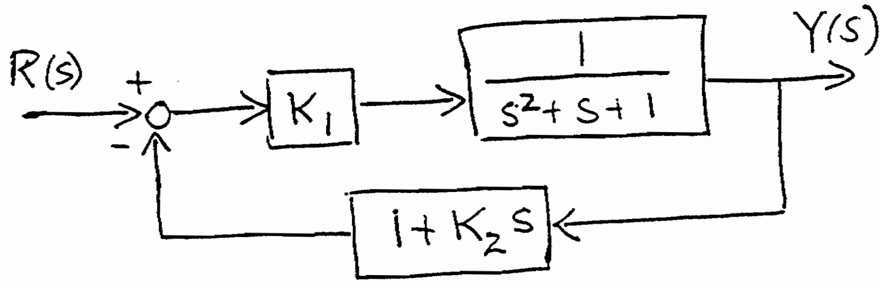
a) Find the differential equation relating input r(t), disturbance v(t) to the output y(t).
b) For K=1, what is the general solution of the homogeneous equation? For K=3, what is it? For K=5, what is it?
c) For K=5, what is the steady state error in responding to a command r(t)=7?
d) For F=5, what is the steady state error produced by the disturbance v(t)=7?
e) What is the general solution for K=5 with both r(t) and v(t) given as 7.
f) For K=5, if r(t)=t what is the steady state rate of growth in error in responding to this command?
g) What time constant is associated with the transients of this system (with K=5).

MECHANICAL ENGINEERING
QUALIFYING EXAMINATION

January 2000

Control

PROBLEM 3



a) Write the plant in stat variable form.
b) Write the feedback and controller outputs in terms of state variables.
c) Write the input/output relationship between r(t) and y(t) in modern state variable form.

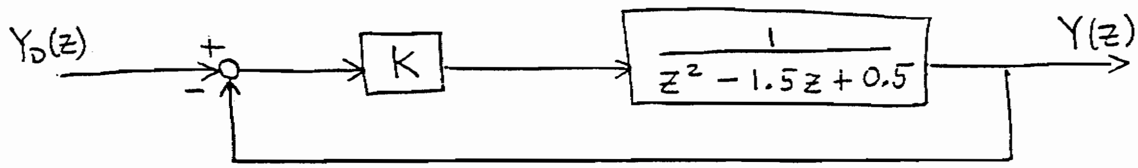
**MECHANICAL ENGINEERING
QUALIFYING EXAMINATION**

January 2000

Control

PROBLEM 4

For the following digital control system with a proportional controller, you want to adjust the controller gain K for good performance.



A) Find the closed loop transfer function from command $Y_D(z)$ to response $Y(z)$.
B) Create the associated difference equation.
C) Write the general solution of the homogeneous equation in terms of REAL VALUED functions, for all gains $K>0$. What range of gains is associated with nonrepeated real roots, with repeated real roots, with nonrepeated complex conjugate roots, complex conjugate repeated roots?
D) Discuss the way that changing the gain K influences the time needed for decay of the transients. Give the time constant as a function of K. Give the settling time as a function of K.
E) If the command $y_D(kT) = 2$, find a particular solution as a function of K. Discuss how changing K influences the accuracy of this part of the solution in following the command.
F) Do the same as Part E but with a command of $y_D(kT) = k$.